Proper Time before Relativity

From Pauli's Phase to Fermat's Principle

Alexander Blum, Jürgen Renn, Don Salisbury

Fritz Haber Institute and Max Planck Institute for the History of Science, Berlin
and Austin College

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Overview

Introduction

The talk begins by examining a strange feature of non-relativistic quantum mechanics, which has been identified as a residue of relativity.

Pursuing this oddity, we find that it is present already in classical physics, and, using variational formulations of mechanics and analogies between optics and mechanics, can even form the starting point for constructing essential elements of special relativity and, invoking the equivalence principle, of general relativity as well.

This provides new observations on the relation between classical and modern physics, and the constraints that classical physics places on its successor theories, both by demanding that insights of classical physics are preserved, and that different domains of classical physics are combined in overarching theoretical frameworks.

Pauli's phase

When Galilei transforming the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$$

the coordinate transformation

$$\begin{aligned} x &\rightarrow x' = x - vt \\ t &\rightarrow t' = t \end{aligned}$$

and thus

$$\psi'(x', t') = \psi(x' + vt', t')$$

is not sufficient.

Pauli's phase

Rather one needs an additional phase transformation, apparently first introduced in Pauli's 1933 Handbuch article "Die Allgemeinen Prinzipien der Wellenmechanik."

$$\psi'(x', t') = \psi(x' + vt', t')e^{-\frac{i}{\hbar}\left(mvx' + \frac{m}{2}v^2 t'\right)}$$

Deriving Pauli's phase

- Start from (coordinate) transformed Schrödinger equation

$$\begin{aligned}\partial_x &= \partial_{x'}\\ \partial_t &= \partial_{t'} - v\partial_{x'}\end{aligned}$$

  Find phase transformation of wave function that conserves form of Schrödinger equation (e.g., Greenberger, 1978).
- Start from (plane wave) solution of untransformed Schrödinger equation, i.e.,

$$\psi(x,t) = e^{-\frac{i}{\hbar}(E(p)t - px)}$$

  Insert new coordinates, and find additional phase transformation needed to give correct expectation values of energy and momentum ($E' = E - pv + mv^2/2$; $p' = p - mv$)in the new reference frame (Pauli).

Origin of Pauli's Phase

When it's not entirely trivialized, the origin of this phase is seen as a relativistic remnant. If one considers the Schrödinger equation as the non-relativistic limit of a Klein Gordon equation, one has to add a constant term mc^2 to the Hamiltonian. For the wave function this means an additional factor of $e^{-\frac{i}{\hbar}mc^2 t}$ - which has no effect (shift of energy by constant).

But when considering the Galilei transform as an approximated Lorentz transform, we have neglected the first correction to the time.

$$t \rightarrow t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} \approx t - \frac{v}{c^2}x + \frac{1}{2}\frac{v^2}{c^2}t$$

Origin of Pauli's Phase

Plugging the usually negligible time transformation

$$t \rightarrow t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} \approx t - \frac{v}{c^2}x + \frac{1}{2}\frac{v^2}{c^2}t$$

into the usually redundant phase factor $e^{-\frac{i}{\hbar}mc^2 t}$, we get an additional factor in the transformed wave function.

$$e^{-\frac{i}{\hbar}mc^2 t'} e^{-\frac{i}{\hbar}\left(mvx' + \frac{m}{2}v^2 t'\right)}$$

...the redundant relativistic phase times Pauli's phase—and all terms of order $1/c^2$ cancel. Greenberger (1979) called (a generalization of) this phase a residue of relativity.

Origin of Pauli's Phase

One can also argue the other way around: In order to eliminate Pauli's phase, one needs to introduce elements of relativity, namely

1. A constant contribution mc^2 to the energy
2. A non-trivial transformation of time under boost transformations

This only eliminates Pauli's phase if we neglect terms of order $1/c^2$—they are introduced by the non-trivial time transformation in the regular phase (without mc^2) and would have to be cancelled by a new phase in order to ensure invariance between inertial frames. Continuing this procedure gives an iterative construction both of the relativistic dispersion relation (1) and of Lorentz transformations (2).

Pauli's phase in classical physics

But there is nothing specifically quantum about this procedure. Remembering that the phase of the one-particle wave function historically originates in the principal function S of Hamilton-Jacobi Theory.

$$H\left(x, \frac{\partial S}{\partial x}\right) + \frac{\partial S}{\partial t} = 0$$

So in our case

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{\partial S}{\partial t} = 0$$

Pauli's phase in classical physics

Under a Galilei transformation, we can again either consider the transformed HJ equation or demand that in the solution

$$S(x,t) = -\frac{1}{2}mv_0^2 t + mv_0 x$$

the integration constant transform as a velocity, i.e., $v_0' = v_0 - v$. In both cases, we have

$$S'(x',t') \neq S(x'+vt',t')$$

but rather

$$S'(x',t') = S(x'+vt',t') - \left(mvx' + \frac{m}{2}v^2 t'\right)$$

A residue of relativity in classical mechanics?

Connection to Relativity

The connection to relativity can be made just as in quantum mechanics. We relate the transformation properties of S to the relativistic transformation properties of t (going to the rest system of the particle).

$$t' \approx t - \frac{v_0}{c^2}x + \frac{1}{2}\frac{v_0^2}{c^2}t = t - \frac{1}{mc^2}S(x,t)$$

The non-relativistic principal function is proportional to the first order relativistic corrections to time in the particle's rest frame.

Connection to Relativity

The more familiar action is the principal function evaluated along a trajectory, in this case $x = v_0 t$ from t_0 to $t_0 + \Delta t$.

$$S \left(= \int_{t_0}^{t_0+\Delta t} L dt \right) = \frac{1}{2} m v_0^2 \Delta t = -mc^2 \left(-\frac{1}{2} \frac{v_0^2}{c^2} \Delta t \right)$$

To be compared with relativistic proper time τ

$$\tau \approx \Delta t - \frac{1}{2} \frac{v_0^2}{c^2} \Delta t = \Delta t - \frac{1}{mc^2} S$$

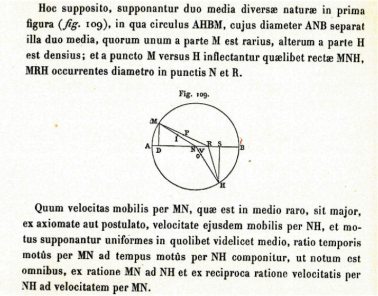
This is not entirely surprising, since the relativistic free particle action is (proportional to) proper time. However, the non-relativistic action is proportional to the first order relativistic corrections to proper time, not simply to its fully non-relativistic limit ($c \to \infty$), which is just coordinate time.

Relativity from Proper Time

- The iterative construction of the relativistic energy-momentum relation and of Lorentz transformations from the elimination of the Pauli phase can thus be motivated more clearly entirely in classical physics from the demand that the action be an invariant measure of time along a trajectory, i.e., from the notion of proper time identified with the classical action.
- It works, because when viewed in this way, the classical action already is a first order relativistic correction and can thus be used as the starting point for an iterative construction.
- The speed of light enters not through electrodynamics, but as a proportionality constant that turns the action into a measure of time.

Fermat and Proper Time

The notion of time along a trajectory is not entirely alien to classical physics.



Hoc supposito, supponantur duo media diversæ naturæ in prima figura (*fig.* 109), in qua circulus AHBM, cujus diameter ANB separat illa duo media, quorum unum a parte M est rarius, alterum a parte H est densius; et a puncto M versus H inflectantur quælibet rectæ MNH, MRH occurrentes diametro in punctis N et R.

Quum velocitas mobilis per MN, quæ est in medio raro, sit major, ex axiomate aut postulato, velocitate ejusdem mobilis per NH, et motus supponantur uniformes in quolibet videlicet medio, ratio temporis motûs per MN ad tempus motûs per NH componitur, ut notum est omnibus, ex ratione MN ad NH et ex reciproca ratione velocitatis per NH ad velocitatem per MN.

Pierre de Fermat (1601–1665)

Maupertuis and Proper Time

But its connection to the action was explicitly denied.

En méditant profondément ſur cette matiere, j'ai penſé que la lumiere, lorſqu'elle paſſe d'un milieu dans un autre, abandonnant déjà le chemin le plus court, qui eſt celui de la ligne droite, pouvoit bien auſſi ne pas ſuivre celui du temps le plus prompt. En effet, quelle préférence devroit-il y avoir ici du temps ſur l'eſpace ? la lumiere ne pouvant plus aller tout à la fois par le chemin le plus court, & par celui du temps le plus prompt, pourquoi iroit-elle plutôt par l'un de ces chemins que par l'autre ? Auſſi ne ſuit-elle aucun des deux;

elle prend une route qui a un avantage plus réel : *le chemin qu'elle tient eſt celui par lequel la quantité d'action eſt la moindre.*

Pierre L. M. de Maupertuis (1698–1759)

Two Optical-Mechanical Analogies

Both light rays and particle paths are determined through variational principles.

- Since for light this can be derived from an underlying wave theory, one can construct a corresponding wave theory of mechanics, completing Hamilton's optical-mechanical analogy. This led de Broglie and Schrödinger to matter waves and wave mechanics.
- Since for light rays the quantity being minimized is a measure of time, the same should hold true for the mechanical action, a requirement that may be called Maupertuis' optical mechanical analogy. In both cases the quantity being minimized is found to be the difference between coordinate and trajectory time (the latter being always zero for light rays). This can form the starting point for constructing special relativity.

Preliminary Conclusions

- The demand of finding an overarching theoretical framework for optics and mechanics (i.e., using optical-mechanical analogies to make the two theories more alike) is such a strong constraint on the development of post-classical physics that constitutive elements of both special relativity and quantum mechanics can be derived from this demand.
- The incorporation of optics with electromagnetism in the 19th century was not a necessary condition for obtaining key insights of modern physics. It rather shifted the focus from seeking an integration of optics and mechanics in terms of variational principles to the role of the ether and the properties of electromagnetic radiation.

Preliminary Conclusions

- The elimination of Pauli's phase from quantum Galilei transformations can form the starting point for an iterative construction of special relativity.
- This can be understood in classical physics as constructing from the classical action an invariant measure of time along a trajectory, replacing absolute coordinate time.
- This works, because the classical action can be taken as the first relativistic correction to proper time, a scalar under changes of inertial frame.
- Historically, this procedure is related to a second optical-mechanical analogy, which was problematized, but never pursued.

Introduction

The second part of this talk generalizes the conclusions of the first part to a generally relativistic context, extending Maupertuis' optical-mechanical analogy with the help of the equivalence principle.

We show that this leads to a derivation of the space-time transformations considered by Einstein in 1912, which can be identified as approximate Rindler transformations.

Combining this with Hamilton's optical-mechanical analogy, an Einsteinian Schrödinger equation is derived by a route alternative to the familiar Kiefer-Singh procedure.

Introduction

The relation of the Einsteinian Schrödinger equation to Hamilton-Jacobi theory and a metric theory of spacetime is examined in more detail.

As the Schrödinger version of the optical-mechanical analogy leads to complex waves, an alternative route is developed, starting from a relativistic Hamilton-Jacobi equation and leading to the Klein-Gordon equation with purely real solutions.

The two optical-mechanical analogies are thus seen to yield basic insights of special and general relativity, as well as two fundamental wave equations, each raising different interpretational problems.

Hamilton-Jacobi implementation of the Equivalence Principle

The free particle Hamilton principal function is

$$S'(\bar{t}, \vec{x'}) = -mc^2\gamma t' + m\gamma\vec{v}\cdot\vec{x'}, \text{ where } \gamma = \left(1 - v^2/c^2\right)^{1/2}.$$

Assume that S' transforms as a scalar under a transformation to an accelerated frame of reference since when evaluated on the particle trajectory it is proportional to the time measured in the clock's rest frame - and therefore invariant.

Hamilton-Jacobi implementation of the Equivalence Principle

Choose as the target $S(t,x) = S'(t'(x,t), t'(x,t)$ the principal function for a non-relativistic particle in a homogeneous gravitational field. This is a Hamilton-Jacobi formulation of Einstein's equivalence principle: The known Hamilton principal function describing a particle in a homogeneous gravitational field is declared to be equivalent the the characteristic function that is obtained via acceleration from an inertial frame. The transformed S is therefore required to satisfy the Hamilton-Jacobi equation

$$mc^2 + \frac{1}{2m}\frac{\partial S}{\partial x_a}\frac{\partial S}{\partial x^a} + mag + \frac{\partial S}{\partial t} = 0. \tag{1}$$

H-J iteration to order c^0 yields new time transformation

As initial spacetime transformations try

$$x' = x + \frac{1}{2}gt^2 \text{ and } t' = t.$$

To order c^0 the form (1) of the Hamilton-Jacobi equation is maintained only if the time transformation is altered to

$$t' = t + \frac{1}{c^2}gxt.$$

We will recognize this as delivering an accelerated frame analogue of Pauli's phase.

H-J iteration to order c^{-2} modifies the H-J equation

Iterating to order $1/c^4$ we must modify the target principal function to satisfy the Hamilton-Jacobi equation

$$mc^2 + mc^2\left(1+\frac{gx}{c^2}\right)\frac{1}{2m}\frac{\partial S}{\partial x_a}\frac{\partial S}{\partial x^a} - \frac{1}{8m^3c^2}\left(\frac{\partial S}{\partial x_a}\frac{\partial S}{\partial x^a}\right)^2 + \frac{\partial S}{\partial t} = 0.$$

The spacetime transformations that produce this form when transforming from an inertial frame are

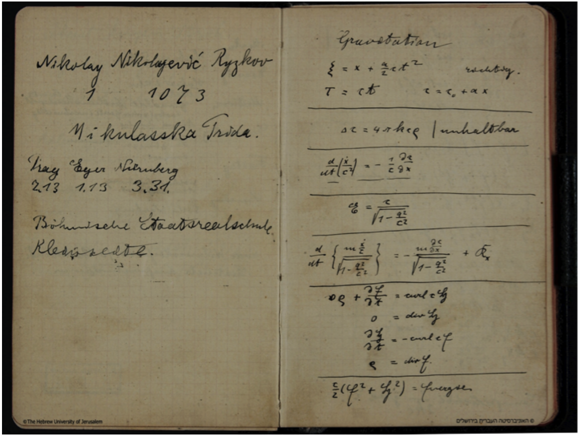
$$t' = t + \frac{1}{c^2}\left(gxt + \frac{1}{6}g^2t^3\right) + \frac{1}{c^4}\frac{1}{6}xg^3t^3,$$

and

$$x' = x + \frac{1}{2}gt^2 + \frac{1}{c^2}\frac{1}{2}xg^2t^2.$$

These are the lowest order contributions to Rindler transformations - also Einstein's transformations of 1912.

Einstein 1912 Interlude



Pages from Einstein's Prague notebook, 1912

Schrödinger's Hamilton-Jacobi route to non-relativistic quantum mechanics

Schrödinger observed that a complex pure phase wave with phase $S/\hbar$ satisfied his wave equation as a consequence of the Hamilton-Jacobi equation (in the limit $\hbar \to 0$),

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + U(\vec{x}) - i\hbar\frac{\partial}{\partial t}\right) e^{iS/\hbar}$$

$$= \left(\frac{1}{2m}\vec{\nabla}S \cdot \vec{\nabla}S \cdot + U(\vec{x}) + \frac{\partial S}{\partial t}\right) e^{iS/\hbar} + \mathcal{O}(\hbar) \to 0 \text{ as } \hbar \to 0.$$

Greenberger's phase arises in an analogous manner to Pauli's phase

The Schrödinger equation that corresponds in this manner to the Hamilton-Jacobi equation (1) is

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + mgx - i\hbar\frac{\partial}{\partial t}\right) e^{iS/\hbar} = 0.$$

Observation 1): The principal function that is obtained under Einstein's transformations up to order c^0 satisfies this Schrödinger equation (28). To this order

$$S(\vec{x}, t) = -m\left(c^2 t + \frac{1}{2}v^2 t - vx - \frac{1}{2}vgt^2 + gxt + \frac{1}{6}g^2 t^3 - v_y y - v_z z\right).$$

Greenberger's phase arises in an analogous manner to Pauli's phase

Observation 2): On the other hand, if the extended Galilean transformation $x' = x + \frac{1}{2}gt^2$ and $t' = t$ of the free principal function is used, the result is

$$S(\vec{x}, t)_{Galileo} = -m\left(c^2t + \frac{1}{2}v^2t - vx - \frac{1}{2}vgt^2 - v_y y - v_z z\right).$$

Concluding observation): The difference (divided by $\hbar$), $-m(gxt + \frac{1}{6}g^2t^3)$, is precisely the term that Greenberger adds to the phase obtained by transforming the free particle Schrödinger equation. The difference arises because the Galilean result does not fully express the proper time.

An Einsteinian Schrödinger equation

The wave function $\Psi = e^{iS/\hbar}$, when S developed to order $1/c^2$, satisfies as a consequence of the Hamilton-Jacobi equation, the Einsteinian Schrödinger equation (to order $\hbar^0$)

$$\left(mc^2 - \frac{\hbar^2}{2m}\nabla^2 + mgx + \frac{1}{c^2}\left(-\hbar^2\frac{gx}{2m}\nabla^2 - \frac{\hbar^4}{8m^3}\nabla^2\nabla^2\right)\right)\psi(\vec{x}, t)$$

$$= i\hbar\frac{\partial\psi(\vec{x}, t)}{\partial t}$$

A pre-Ehrenfest theorem

We can construct pure phase solutions of the Einsteinian Schrödinger of the form

$$\Psi_\alpha(\vec{x},t) = e^{i(\int \frac{\partial S(\vec{x},t;\alpha)}{\partial t}dt + \int \vec{\nabla}S(\vec{x},t;\alpha)\cdot d\vec{x})/\hbar},$$

where the constants α deliver a complete solution of the principal function S. Then we identify a generally variable angular frequency

$$\omega(\vec{x},t;\alpha) = -\frac{1}{\hbar}\frac{\partial S}{\partial t},$$

and a variable wave number vector

$$\vec{k}(\vec{x},t;\alpha) = \frac{1}{\hbar}\vec{\nabla}S.$$

Thus the Hamilton-Jacobi equation is reinterpreted as a wave dispersion relation

$$\omega(\vec{x},t;\alpha) = \frac{1}{\hbar}H\left(\vec{x},\hbar\vec{k}(\vec{x},t;\alpha)\right).$$

A pre-Ehrenfest theorem

Thus if we build an appropriate superposition of the states over α that peaks at $\vec{k}_0$ we can construct a wave packet that moves along the classical spacetime trajectory. This follows from the classical dynamical equation $\frac{dx^a}{dt} = \frac{\partial H}{\partial p_a}$ since the group velocity is

$$\frac{\partial \omega}{\partial k_a}\Big|_{\vec{k}_0} = \frac{\partial H(\vec{x},\vec{p})}{\partial p_a}\Big|_{\vec{p}=\hbar\vec{k}_0} .$$

Appropriately constructed wave packet solutions of the Einsteinian-Schrödinger equation therefore move along the correct classical spacetime trajectories.

Extension to Rindler spacetime

The full Rindler transformations to a rigidly accelerated frame of reference are

$$x' = \left(\frac{c^2}{g} + x\right) \cosh\left(\frac{gt}{c}\right) - \frac{c^2}{g},$$

and

$$ct' = \left(\frac{c^2}{g} + x\right) \sinh\left(\frac{gt}{c}\right).$$

Under these transformations we read off from the transformed Hamilton-Jacobi equation that the Hamiltonian is

$$H(\vec{x}, \vec{p}) = -mc^2 \left(1 + \frac{gx}{c^2}\right) \left(1 + \frac{p^2}{m^2 c^2}\right)^{1/2}$$

Rindler proper time and metric

This Hamiltonian corresponds to a maximized proper time increment

$$d\tau = \left(\left(1 + \frac{gx}{c^2}\right)^2 - \frac{v^2}{c^2} \right)^{1/2} dt,$$

i.e., one nontrivial metric component

$$g_{00} = -\left(1 + \frac{gx}{c^2}\right)^2 .$$

Klein-Gordon equation

Consider a real-valued wave, $\sin(\epsilon S)$, where ϵ is a constant with dimension one over an action.

Theorem: This wave satisfies the Klein-Gordon equation on the Rindler background in the limit of large ϵ as a consequence of the Hamilton-Jacobi equation!

Proof: Keeping only ϵ^2 terms, we find

$$\left(g^{\mu\nu}\nabla_\mu\nabla_\nu - \epsilon^2 m^2 c^2\right)\sin(\epsilon S)$$

$$= -\left(1+\frac{gx}{c^2}\right)^{-2}\frac{\epsilon^2}{c^2}\left(\frac{\partial S}{\partial t} - H(\vec{x},\vec{\nabla}S)\right)\left(\frac{\partial S}{\partial t} + H(\vec{x},\vec{\nabla}S)\right)\sin(\epsilon S)$$

$$= 0.$$

Back to Jacobi, Hamilton, and Fermat

We have found our way back to a comprehensive purely classical implementation that unifies the wave and particle treatments of both massive and massless particles, provided that a relativistic framework as suggested by Maupertuis' optical-mechanical analogy is accepted. When $m = 0$ we can construct a (scalar) light wave packet that satisfies Fermat's principle. When $m \neq 0$ we have wave packets that move along the correct relativistic trajectories. This is a program that could have in principle been undertaken by Jacobi or even Hamilton if they had had reason to take Maupertuis' optical-mechanical analogy seriously and to believe in proper time, for instance if muons had been discovered earlier.