Time and Observables in General Relativity

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Plan of Talk

1. Review of canonical coordinate-transformation-induced symmetry group
2. Classical intrinsic time gauge fixing
3. Proposed generalized time-dependent Schrödinger in quantum cosmology
4. Semi-classical limit

1 - Canonical symmetry group

The dynamical model

Isotropic cosmology

$$g_{\mu\nu} = l_P^2 \begin{pmatrix} -N^2 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & a^2 \end{pmatrix}$$

Lagrangian with massless scalar field $\sqrt{\frac{m_P}{t_P}}\phi$

$$L = h\left(-\frac{3a\dot{a}^2}{4\pi N} + \frac{a^3\dot{\phi}^2}{2N} \right) \qquad H = \frac{N}{h}\left(-\frac{\pi p_a^2}{3a} + \frac{p_\phi^2}{2a^3} \right) + \lambda p_N$$

Constraints

$$-\frac{\pi p_a^2}{3a} + \frac{p_\phi^2}{2a^3} \approx 0 \qquad p_N \approx 0$$

The symmetry group

Infinitesimal time transformations are projectable under the Legendre map

(functions of $\dot{N}$ are not projectable)

$$t' = t - N^{-1}\xi \Rightarrow \bar{\delta} N = \dot{N} N^{-1}\xi + N\frac{d}{dt}\left(N^{-1}\xi\right) = \dot{\xi}$$

Symmetry is a transformation group on the variables *a*, *N*, and their canonical momenta

Canonical generator of infinitesimal symmetry transformations

$$\xi\left(-\frac{\pi p_a^2}{3a} + \frac{p_\phi^2}{2a^3}\right) + \dot{\xi} p_N$$

2. Intrinsic Coordinates

General solutions

$$a(t) = \left(N_0 t + \int_0^t dt' \int_0^{t'} dt'' \lambda(t'') + a_0^3 \right)^{\frac{1}{3}} \qquad \phi(t) = \phi_0 \pm \sqrt{\frac{1}{6\pi}} \ln\left(\frac{N_0 t + \int_0^t dt' \int_0^{t'} dt'' \lambda(t'') + a_0^3}{a_0^3} \right)$$

$$N(t) = N_0 + \int_0^t dt' \lambda(t')$$

Choose intrinsic time T

$$T = a^2(t) \Rightarrow \left(N_0 t + \int_0^t dt' \int_0^{t'} dt'' \lambda(t'') + a_0^3 \right) = (T)^{\frac{3}{2}}$$

Approach 1: transform to intrinsic coordinates

Claim: $\phi(t(T))$ and $N(t(T))\frac{dt}{dT}$

are invariant under the canonical action of the canonical symmetry group

$$\phi(T) = \phi_0 \pm \sqrt{\frac{1}{6\pi}}\left(\frac{3}{2}\ln T - 3\ln a_0\right) = \phi(t) \pm \sqrt{\frac{1}{6\pi}}\left(\frac{3}{2}\ln T - 3\ln a(t)\right)$$

$$N(T) = N(t(T))\frac{dt}{dT} = \frac{3T^{\frac{1}{2}}}{2}$$

Approach 2: gauge transform to solutions satisfying intrinsic coordinate choice

Generator of finite gauge transformation

$$V_\xi(s,t) = \exp\left(s\{-, G_\xi(t)\}\right)$$

$$a_\xi(s,t) = a_0\left(N_0 t + \int_0^t dt' \int_0^{t'} dt'' \lambda(t'') + s\xi(t) + a_0^3\right)^{\frac{1}{3}}$$

$$N_\xi(s,t) = N(t) + s\dot{\xi}(t)$$

$$\phi_\xi(s,t) = \phi_0 \pm \sqrt{\frac{1}{6\pi}} \ln\left(\frac{N_0 t + \int_0^t dt' \int_0^{t'} dt'' \lambda(t'') + s\xi(t) + a_0^3}{a_0^3}\right)$$

Solve $t = a_\xi^{\,2}(1,t)$ for ξ and substitute into other variables

Approach 3: impose gauge conditions

$$t = a^2(t)$$

Preservation in time leads to new condition

$$N + \frac{3h}{4\pi p_a} = 0$$

Dirac procedure yields following equations of motion

$$\dot{N} = -\frac{3h}{8\pi a^2 p_a} \qquad \dot{a} = \frac{1}{2a}$$

$$\dot{\phi} = -\frac{3p_\phi}{4\pi a^3 p_a} \qquad \dot{p}_\phi = 0$$

Approach 4: solve constraints to eliminate a and p_a

The independent variables ϕ and p_ϕ are governed by the gauge fixed Hamiltonian

$$H_{GF} = \frac{p_\phi^2}{4\mathrm{h}t}$$

3. Generalized time-dependent Schrödinger equation

Use discrete time eigenvalues from loop gravity

$$t_k = \frac{k}{6}, \quad k = 0,1,2, \mathrm{K}$$

$$\text{Let } |\psi(\phi, t_{k+1})\rangle = \left(1 - \frac{i\Delta t H}{\mathrm{h}}\right) |\psi(\phi, t_k)\rangle = \left(1 - \frac{9i}{2\mathrm{h}^2(k+1)}\right) |\psi(\phi, t_k)\rangle$$

4. Semi-classical limit

Take initial state to be minimum uncertainty wave packet

$$\psi(\phi, t_0) = c \exp\left[-\frac{(\phi - \phi_0)^2}{4\sigma^2} + i\frac{p_0}{\hbar}\right]$$

Then it follows that

$$\langle \phi(t_0 + \Delta t) \rangle = \phi_0 + \frac{3p_0}{2\hbar t_0}\Delta t$$