A Generalized Schrödinger Equation for Loop Quantum Cosmology

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Plan of Talk

1. Review of canonical coordinate-transformation-induced symmetry group
2. Classical intrinsic time gauge fixing
3. Proposed generalized time-dependent Schrödinger in quantum cosmology
4. Semi-classical limit

Collaborators and references

- "Reparameterization invariants for anisotropic Bianchi I cosmology with a massless scalar source (with J Helpert and A Schmitz), gr-qc/0503014
- "The issue of time in generally covariant theories and the Komar-Bergmann approach to observables in general relativity," (with J Pons) Phys.Rev.D71, 12402 (2005) gr-qc/0503013
- "The gauge group in the Ashtekar-Barbero formulation of canonical gravity,", in Proceedings of the Ninth Marcel Grossmann Meeting, edited by V.G. Gurzadyan, R. T. Jantzen and R. Ruffini, (World Scientific, New Jersey, 2002), 1298 (with J. Pons)
- "The gauge group and the reality conditions in Ashtekar's formulation of general relativity," Phys. Rev. **D62** , 064026 (2000) (with J.M. Pons and L.C. Shepley)
- "The gauge group in the real triad formulation of general relativity," Gen. Rel. Grav. **32**, 1727 (2000) (with J.M. Pons and L.C. Shepley)
- "Gauge transformations in Einstein-Yang-Mills theories," J. Math. Phys. **41**, 5557 (2000) (with J.M. Pons and L.C. Shepley)
- "The realization in phase space of general coordinate transformations," Phys. Rev. D27, 740 (1983) (with K. Sundermeyer)

1 - Canonical symmetry group

The dynamical model

Isotropic cosmology

$$g_{\mu\nu} = \begin{pmatrix} -N^2 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & a^2 \end{pmatrix}$$

Lagrangian with massless scalar field ϕ

$$L = -\frac{6a^3}{\kappa N} + \frac{a^3\dot{\phi}^2}{2N} \qquad H = -\frac{\kappa N p_a^2}{24a} + \frac{N p_\phi^2}{2a^3} + \lambda p_N$$

Constraints

$$-\frac{\kappa p_a^2}{24a} + \frac{p_\phi^2}{2a^3} \approx 0 \qquad p_N \approx 0$$

The symmetry group

Infinitesimal time transformations are projectable under the Legendre map

(functions of $\dot{N}$ are not projectable)

$$t' = t - N^{-1}\xi \Rightarrow \bar{\delta} N = \dot{N} N^{-1}\xi + N \frac{d}{dt}\left(N^{-1}\xi\right) = \dot{\xi}$$

Symmetry is a transformation group on the variables *a, N,* and their canonical momenta

Canonical generator of infinitesimal symmetry transformations

$$\xi\left(-\frac{\kappa p_a^2}{24a} + \frac{p_\phi^2}{2a^3}\right) + \dot{\xi} p_N$$

2. Intrinsic Coordinates

General solutions

$$a(t) = a_0 \left(\frac{N_0 t + \int_0^t dt' \int_0^{t'} dt'' \lambda(t'') + \ell^{-2} a_0^3}{\ell^{-2} a_0^3} \right)^{\frac{1}{3}} \qquad \phi(t) = \phi_0 + \sqrt{\frac{2}{3\kappa}} \ln \left(\frac{N_0 t + \int_0^t dt' \int_0^{t'} dt'' \lambda(t'') + \ell^{-2} a_0^3}{\ell^{-2} a_0^3} \right)$$

$$N(t) = N_0 + \int_0^t dt' \lambda(t')$$

Choose intrinsic time T

$$T = \ell^{-2} a^2(t) \Rightarrow \left(\frac{N_0 + \int_0^t dt' \int_0^{t'} dt'' \lambda(t'') + \ell^{-2} a_0^3}{\ell^{-2} a_0^3} \right) = \left(\frac{\ell^2 T}{a_0^2} \right)^{\frac{3}{2}}$$

Approach 1: transform to intrinsic coordinates

Claim: $\phi\left(t(T)\right)$ and $N\left(t(T)\right)\dfrac{dt}{dT}$

are invariant under the canonical action of the canonical symmetry group

$$\phi(T) = \phi_0 + \frac{3}{2}\sqrt{\frac{2}{3\kappa}}\ln\left(\frac{\ell^2 T}{a_0^2}\right) = \phi(t) + \frac{3}{2}\sqrt{\frac{2}{3\kappa}}\ln\left(\frac{\ell^2 T}{a^2(t)}\right)$$

$$N(T) = N\left(t(T)\right)\frac{dt}{dT} = \ell T^{\frac{1}{2}}$$

Approach 2: gauge transform to solutions satisfying intrinsic coordinate choice

Generator of finite gauge transformation

$$V_\xi(s,t) = \exp\left(s\left\{-,G_\xi(t)\right\}\right)$$

$$a_\xi(s,t) = a_0\left(\frac{N_0 t + \int_0^t dt' \int_0^{t'} dt'' \lambda(t'') + s\xi(t) + \ell^{-2} a_0^3}{\ell^{-2} a_0^3}\right)^{\frac{1}{3}} \qquad N_\xi(s,t) = N(t) + s\dot{\xi}(t)$$

$$\phi_\xi(s,t) = \phi_0 + \sqrt{\frac{2}{3\kappa}} \ln\left(\frac{N_0 t + \int_0^t dt' \int_0^{t'} dt'' \lambda(t'') + s\xi(t)\ell^{-2} a_0^3}{\ell^{-2} a_0^3}\right)$$

Solve $t = \ell^{-2} a_\xi{}^2(1,t)$ for ξ and substitute into other variables

Approach 3: impose gauge conditions

$$t = \ell^{-2} a^{2}(t)$$

Preservation in time leads to new condition

$$N + \frac{6\ell^2}{\kappa p_a} = 0$$

Dirac procedure yields following equations of motion

$$\dot{N} = -\frac{3\ell^4}{\kappa a^2 p_a} \qquad \dot{a} = \frac{\ell^3}{2a}$$

$$\dot{\phi} = -\frac{6\ell^2 p_\phi}{\kappa a^3 p_a} \qquad \dot{p}_\phi = 0$$

Approach 4: solve constraints to eliminate a and p_a

$$\Rightarrow \dot{N} = \frac{3\ell}{4t^{\frac{1}{2}}}$$

The independent variables ϕ and p_ϕ are governed by the gauge fixed Hamiltonian

$$H_{GF} = \sqrt{\frac{3}{\kappa}}\frac{p_\phi}{t}$$

3. Generalized time-dependent Schrödinger equation

Use discrete time eigenvalues from loop gravity

$$t_k = \frac{\ell_P k}{6}, \quad k = 0?,\ 1,\ 2,\ \ldots$$

Let $\left|\psi\left(\phi, t_{k+1}\right)\right\rangle = \left(1 - \frac{i\ell_P}{6\hbar}\right)\left|\psi\left(\phi, t_k\right)\right\rangle$

4. Semi-classical limit

Take initial state to be minimum uncertainty wave packet

$$\psi(\phi, t_0) = c \;\; \exp\left[-\frac{(\phi-\phi_0)^2}{4\sigma^2} + i\frac{p_0}{\hbar}\right]$$

Then it follows that

$$\psi(\phi, t_0 + \Delta t) \approx c \;\; \exp\left[-\frac{\left(\phi - \phi_0 - \sqrt{\frac{3}{\kappa}}\frac{\Delta t}{t_0}\right)^2}{4\sigma^2} + i\frac{p_0}{\hbar}\right]$$