An intrinsic Hamilton-Jacobi approach to general relativity and its Syracuse school origins

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Overview

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1. INTRODUCTION

Introduction

Focus of this talk:

- The evolving work over two decades in Syracuse, New York, by Peter Bergmann and his associates that led to the notion of general relativistic reduced phase space whose elements were full four-diemsional diffeomorphism invariants
- The relationship of this work to later Hamilton-Jacobi approaches to general relativity

Some questions to be addressed:

- How did the recognition emerge that the realization of general covariance in phase space necessitated metric field dependence of the transformation group?
- What is the relation between the modern geometric differential description and Hamilton-Jacobi approaches?

2. BRIEF HISTORY OF HAMILTONIAN GENERAL COVARIANCE AT SYRACUSE

Rosenfeld's 1930 tetrad gravitational Lagrangian

"Zur Quantelung der Wellenfelder", *Annalen der Physik* **397**, 113 (1930) Translation by Salisbury and Sundermeyer [Rosenfeld, 2017]

$$\mathcal{L} = \frac{1}{2\kappa}(-g)^{\frac{1}{2}} E^{\mu}_{I} E^{\nu}_{J} \left({\omega_{\mu}}^{I}{}_{L} {\omega_{\nu}}^{LJ} - {\omega_{\nu}}^{I}{}_{L} {\omega_{\mu}}^{LJ} \right)$$
$$+\Re\left\{ (-g)^{1/2} \left[\frac{1}{2} i \bar{\psi} \gamma^{\mu} \left(\overrightarrow{\partial}_{\mu} + \Omega_{\mu} \right) \psi - m \bar{\psi} \psi \right] \right\} + \mathcal{L}_{em}$$

Tetrads E^{μ}_{I}, Rotation coefficients ${\omega_{\mu}}^{I}{}_{L}$, Fermion field ψ, spinor connection $\Omega_{\mu} = \frac{1}{4}\gamma^{I}\gamma^{J}\omega_{\mu IJ}$.

Rosenfeld's 1930 tetrad Hamiltonian density prehistory!

Rosenfeld invented a systematic procedure for solving for the velocities $\dot{E}_I^\mu$ in terms of the conjugate momenta given that the Jacobian matrix $\frac{\partial^2 \mathcal{L}}{\partial \dot{E}_I^\mu \dot{E}_J^\nu}$ is singular. Although he did not do this explicitly for this model, the result (see [Salisbury & Sundermeyer, 2017]) is

$$\mathcal{H} = \mathcal{H}_0 \left[g_{ab}, p^{ab}, A_a, p^a, \psi, \psi^\dagger \right] + \lambda_I \mathcal{F}^I + \lambda_{IJ} \mathcal{F}^{[IJ]} + \lambda \mathcal{F}$$

where $\mathcal{F}^I$, $\mathcal{F}^{[IJ]}$ and $\mathcal{F}$ are primary constraints and λ_I, λ_{IJ} and λ are arbitrary spacetime functions.

Preceeding Bergmann and Dirac by twenty years! See [Salisbury, 2009].

Rosenfeld's infinitesimal phase space symmetry generator

Rosenfeld proved that the vanishing Noether charge generated the correct variations of all of the phase space variables under all of the local symmetries. Most importantly for us is that the active variations under the infinitesimal coordinate transformations $x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$ are correct.

His conserved and vanishing generating density is

$$-\mathcal{F}^I e_{0I}\dot{\xi}^0 - \mathcal{F}^I e_{aI}\dot{\xi}^a - \mathcal{F} A_0 \dot{\xi}^0 - p^{aI} e_{\nu I}\xi^{\nu}_{,a} - p^a A_{\nu}\xi^{\nu}_{,a} - \mathcal{H} A_0 \xi^0 - \mathcal{G}_a \xi^a$$

$$-\mathcal{F}\dot{\xi} + p^a \xi_{,a} + i\frac{e}{\hbar c} p_{\psi}\psi\xi - i\frac{e}{\hbar c} p_{\psi^{\dagger}}\psi^{\dagger}\xi + \mathcal{F}_{[IJ]}\xi^{IJ} = 0$$

Peter Bergmann and Paul Dirac

We leap two decades forward to the contributions to constrained Hamiltonian dynamics of Peter Bergmann and Paul Dirac - beginning in 1949.

Dirac never concerned himself with the phase space realization of the full general covariance group. See his Vancouver lectures, [Dirac, 1950] [Dirac, 1951]

[Bergmann, 1949], later with Jim Anderson [Anderson & Bergmann, 1951] and numerous collaborators including [Goldberg, 1953] did concern themselves with this symmetry. In particular a joint publication with Ralph Schiller [Bergmann & Schiller, 1953] explicitly employed the vanishing Noether charge.

The origin of the vanishing Noether charge

Assume that the Lagrangian density $\mathfrak{L}$ plus a term linear in second derivatives $S^\mu = f^{A\mu\rho}(y)y_{A,\rho}$ transforms under infinitesimal coordinate transformations $x'^\mu = x^\mu + \xi^\mu(x)$ as a scalar density of weight one, yielding an identity

$$\bar{\delta}\mathfrak{L} \equiv -\left(\bar{\delta}S^\mu\right)_{,\mu} + \left(\left(\mathfrak{L} + S^\nu_{,\nu}\right)\xi^\mu\right)_{,\mu} = \mathfrak{L}^A\bar{\delta}y_A + \left(\frac{\partial\mathfrak{L}}{\partial y_{A,\mu}}\bar{\delta}y_A\right)_{,\mu}$$

$\mathfrak{L}^A = 0$ are the Euler-Lagrange field equations for the fields $y_A(x)$.

The origin of the vanishing Noether charge and the Anderson Bergmann generator

Then when the field equations are satisfied, defining

$$\mathfrak{C}^\mu := -\bar{\delta} S^\mu + \left(\mathfrak{L} + S^\nu_{,\nu}\right) \xi^\mu - \frac{\partial \mathfrak{L}}{\partial y_{A,\mu}} \bar{\delta} y_A,$$

$$\frac{d}{dt} \int d^3x \mathfrak{C}^0 = 0.$$

[Anderson & Bergmann, 1951]

$$0 = \mathfrak{C}^0 = {}^0A_\mu \xi^\mu + {}^1A_\mu \dot{\xi}^\mu$$

They showed that the ${}^1A_\mu$ were primary constraints.

The diffeomorphism Lie algebra puzzle

Anderson and Bergmann proved that no higher time derivatives of the descriptors could appear - even when considering nested commutators of this generator.

Yet nested commutators of the diffeomorphism Lie algebra

$$\xi^{\mu}_{1,\nu}\xi^{\nu}_{2} - \xi^{\mu}_{2,\nu}\xi^{\nu}_{1}$$

do yield higher time derivatives. Does one therefore have a realization of diffeomorphism covariance?

Dirac's Hamiltonian and Bergmann's interpretation

[Dirac, 1958] proposed that the proper candidates for gravitational phase space variables should have the property that their transformations under spacetime coordinate changes do not depend on time derivatives of the descriptors ξ^μ.
[Bergmann, 1962] deduced that the descriptors involved a

compulsory metric field dependence,

$$\xi^\mu = n^\mu \epsilon^0 + \delta^\mu_a \epsilon^a$$

where $n^\mu = \left(N^{-1}, -N^{-1}N^a\right)$ is the normal to the spacelike hypersurface. N and N^a are the lapse and shift components of the metric

$$g_{\mu\nu} = \begin{pmatrix} N^2 - g_{ab}N^a N^b & g_{ab}N^b \\ g_{ab}N^b & g_{ab} \end{pmatrix}.$$

The Bergmann - Komar group symmetry group

Bergmann also calculated the resulting commutator algebra,

$$\epsilon^{\rho} = \delta^{\rho}_{a}\left(\epsilon^{a}_{1,b}\epsilon^{b}_{2} - \epsilon^{a}_{2,b}\epsilon^{b}_{1}\right) + e^{ab}\left(\epsilon^{0}_{1}\epsilon^{0}_{2,b} - \epsilon^{0}_{2}\epsilon^{0}_{1,b}\right) + n^{\rho}\left(\epsilon^{a}_{2}\epsilon^{0}_{1,a} - \epsilon^{a}_{1}\epsilon^{0}_{2,a}\right).$$

e^{ab} is the inverse of the 3-metric g_{ab}.

[Bergmann & Komar, 1972] interpreted this algebra as representing a compulsory non-local metric-dependent transformation group.

3. GENERAL PHASE SPACE TRANSFORMATIONS OF FIELD SOLUTIONS

Bergmann and Schiller - 1953

[Bergmann & Schiller, 1953] identified the form of the tangent bundle functional $f_A\left(y_B, y_{C,\mu}\right)$ that could be translated into a phase space variation $\delta y_A\left(y_B, \pi^C\right)$ that transformed to new solutions of Einstein's equations.

They argued that the relevant physically meaningful phase space was the quotient space of the larger group modulo the diffeomorphism invariance subgroup. They called this the reduced phase space.

Hamilton-Jacobi variations of the action

Hamilton-Jacobi variations with an eye toward quantization were first undertaken in field theory by [Weiss, 1936], inspired by the [Cartan, 1922] treatment of the finite dimensional case. Weiss defined a "complete variation" in which both the fields on the initial and final spacelike hypersurfaces , as well as the hypersurfaces themselves are independently varied, requiring that all variations are to new solutions. The net variation is then

$$\delta_0 y_A(x) =: y_A(x) - y_{A,\mu}(x)\delta x^\mu(x).$$

The corresponding varied action about solutions is then

$$\delta S = \int \left[\tilde{\pi}^A \delta y_A - \tilde{\mathcal{H}}\delta t - \tilde{\mathcal{P}}_a \delta x^a\right] d^3x \,\Big|_{t_2}^{t_1},$$

where the 'tilde' signifies that the momenta are to be considered tangent space functionals.

Relativistic free particle example

Consider the reparameterization covariant action for a free particle of unit mass, in units in which $c = 1$,

$$S = \int (-\dot{q}^2)^{1/2} d\theta = \int L d\theta,$$

Then

$$dS = \frac{\partial L}{\partial \dot{q}^\mu} dq^\mu - \left(\frac{\partial L}{\partial \dot{q}^\mu} \dot{q}^\mu - L \right) d\theta =: \tilde{p}_\mu dq^\mu - \tilde{H}(\dot{q}) d\theta,$$

where $\tilde{H} = (-\dot{q}^2)^{1/2} \left(\tilde{p}^2 + 1 \right)$ is constrained to vanish.

Reparameterization covariance

However, before one makes this choice the Hamiltonian model is reparameterization covariant - but under the infinitesimal parameter change $\theta' = \theta - (-\dot{q}^2)^{-1/2}\xi(\theta)$. (The factor $(-\dot{q}^2)^{-1/2}$ is the analogue of Bergmann's required lapse function.) Indeed, substitution of this variation with the corresponding variation of q^μ into the variation of the action yields the correct generator of this variation,

$$C(\epsilon) = \frac{\xi}{2}\left(p^2 + 1\right).$$

Particle invariants and intrinsic time evolution

The reparameterization generator may deployed to transform the particle solutions in any parameterization θ' to solutions satisfying the gauge condition $q^0 = \theta$. The general solution is a power series in θ whose coefficients are reparameterization invariants. The invariant observables associated with q^a are (see [Pons *et al.* , 2009])

$$q^a(\theta) = q^a(\theta') - \frac{p^a}{p^0} q^0(\theta') + \frac{p^a}{p^0}\theta$$

The conventional particle H-J approach

The free particle symplectic potential becomes with this gauge choice

$$dS = -H(p_b) + p_a dq^a = -\left(p_a p^a + 1\right)^{1/2} d\theta + p_a dq^a$$

The general particle solutions can obtained by finding the complete solution of the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial \theta} + H\left(\frac{\partial S}{\partial q^b}\right) = 0.$$

The complete solutions $S(q^a, \theta; P_b)$ depend on the three constants P_b.

Application to general relativity

Although a form of the symplectic potential has been used to advantage, in particular by [Lee & Wald, 1990], they did not derive from it the general form of the generator of diffeomorphism symmetries. Their work was actually the stimulus of my work with Pons and Shepley [Pons *et al.* , 1997] in which we proved that the decomposition of infinitesimal coordinate transformations in terms of tangent and normal variations was required in order to project the tangent space variations onto phase space (the cotangent space).

We actually derived the resulting generator through group theoretical arguments.

The complete diffeomorphism generator

The complete generator of diffeomorphism-related transformations general relativity is

$$\mathcal{G}_\epsilon(t) = P_\mu \dot{\xi}^\mu + (\mathcal{H}_\mu + \int d^3x' \int d^3x'' N^{\rho'} C^{\nu''}_{\mu\rho'} P_{\nu''})\xi^\mu.$$

where

$$\{\mathcal{H}_\mu(x), \mathcal{H}_\rho(x')\} = C^{\nu''}_{\mu\rho'}\,[g_{ab}]\,\mathcal{H}_{\nu''}$$

A claim and a puzzle

I claim that this generator can actually be obtained through the substitution of variations of the metric fields directly into the symplectic potential - provided one retains the lapse and shift.

This is actually in agreement with the fact that when vector arguments of the symplectic two-forms are null vectors this corresponds to the fact that these vectors are obtained through the action of the gauge symmetry group.
There are two puzzles. The first is the question why Bergmann abandoned the lapse and shift as canonical variables. His original retained all the variables, and did contain time derivatives of the descriptors. The second is: why didn't Pons, Salisbury, and Shepley make this substitution?

Note that the lapse and shift must be retained in order to arbitrarily alter the time foliation.

4. AN INTRINSIC HAMILTON-JACOBI APPROACH TO GENERAL RELATIVITY

First reconsider the free particle

The choice $\theta = q^0$ was an example of a choice of intrinsic time. In the case of general relativity we have proven that candidates must be spacetime scalars. In this reparameterization covariant model they must be reparameterization scalars. In this choice was one of many possibilities.
We could choose the proper time as the intrinsic time through a judicious canonical transformation of the symplectic potential. Choose as the time variable $T = -q^0/p_0$. Make a canonical change of variables by finding a generating function $F(q^0, T)$ such that $p_0 dq^0 = PdT + + \frac{\partial F}{\partial q^0} dq^0 + \frac{\partial F}{\partial T} dT$. The it turns out that the new generator of projectable reparameterizations is

$$G(\epsilon) = \xi(P + \ln T) + \frac{\xi}{2}\left(p_a p^a + 1\right) = 0.$$

Time evolution versus diffeomorphisms

This can be solved so as to eliminate both T and P, ultimately yielding the transformed symplectic form
$dS = \left[-\frac{1}{2}\left(p^a p_a + 1\right) + \ln(\theta)\right] d\theta + p_a dq^a$.
Our proposal is that similar canonical transformations to intrinsic spacetime coordinates can be carried out in general relativity

An implementation of Rovelli's partial variable program

Now that we have the full diffeomorphism group at our disposal, we can employ it to establish correlations between partial variables. One possible implementation, in principle, is to locate temporal and spatial landmarks by referring to curvature even in the vacuum case. There are of course many more possibilities when matter is present. We will employ these landmarks as "intrinsic" coordinates. Such coordinates must be formed from spacetime scalars. Thus we choose $X^\mu[g_{ab}, p^{cd}]$.

In the vacuum case we propose the use of the four Weyl curvature scalars, as originally suggested by [Komar, 1958]. They are quadratic and cubic in the Weyl tensor.
[Bergmann & Komar, 1960] showed that they are expressible solely in terms of the three metric and its conjugate momenta.

Non-trivial classical evolution with spatial landmarks

The resulting evolution is with respect to partial variables - in the sense of [Rovelli, 1991].

There is also obvious variation in spatial directions. There is more to space than topology!

We shall see that having selected intrinsic coordinates the evolution is unique with regard to the in principle measureable curvature coordinates. The behavior is insensitive to whatever coordinates one employed before transforming to intrinsic coordinates. But on the other hand, given any initial intrinsic coordinates one can undertake arbitrary changes in these new intrinsic coordinates - yielding physically distinguishable evolutions. Thus we have a paradoxical situation where we are dealing with general coordinate invariants, and yet we can meaningfully arbitrarilly alter the intrinsic coordinate choices. Rovelli has referred to this phenomena as involving evolving constants of the motion.

Intrinsic coordinate gauge conditions

We choose intrinsic coordinates through the gauge conditions $\theta^\mu = X^\mu[g_{ab}, p^{ab}]$. Given any solution trajectory in phase space we can then determine the phase space dependent finite descriptors $\xi^\mu[g_{ab}, p^{ab}] := \xi^\mu[y]$ that will gauge transform these solutions to those that satisfy the gauge conditions.

The explicit construction of evolving constants of the motion

This construction yields Taylor expansions in the coordinates θ^μ - now themselves diffeomorphism invariants. The coefficients in the Taylor expansions are functionals of g_{ab} and p^{ab} that are explicitly diffeomorphism invariants. This applies also to the invariant lapse and shift.

$$\mathcal{I}_\phi = \sum_{n_\mu=0}^{\infty} \frac{1}{n_0!\, n_1!\, n_2!\, n_3!} (\theta^0)^{n_0} (\theta^1)^{n_1} (\theta^2)^{n_2} (\theta^3)^{n_3}\, \mathcal{C}_{n_0,n_1,n_2,n_3}[g_{ab}, p^{ab}]$$

Kuchar-inspired canonical transformations

Canonical transformations can in principle be carried out to new canonical variables including X^μ and canonical conjugates π_μ - but without imposing gauge conditions. The theory in terms of these new variables is still fully diffeomorphism covariant - with corresponding Hamiltonian constraints. Each choice yields a new form for the constraints and a new Hamilton-Jacobi equation with a corresponding "natural" choice of temporal and spatial partial variables - with the scalar constraint now expressed in terms of the X^μ.

This "natural" choice is the one that results through the solutions of the Hamilton-Jacobi equation.

Free relativistic particle example

Choose as the intrinsic evolution parameter the proper time. This corresponds to a canonical change of $T = -m\frac{q^0}{p_0}$, and our task is to find the canonical generating function $G(q^0, T)$ such that the simplectic one-form contribution $p_0 dq^0$ becomes

$$PdT + \frac{\partial G}{\partial q^0}dq^0 + \frac{\partial G}{\partial T}dT.$$

Having made the canonical change of variables, we of not yet made a choice of an intrinsic time. The rewritten mass shell constraint still generates arbitrary infinitesimal reparamtereizations of the form $\theta' = \theta - \left(-\dot{q}^2\right)^{-1/2}\xi(\theta)$. This change is in fact generated by the transformed mass shell constraint, with the generator taking the form

$$0 = \xi\left(P + \ln(T)\right) + \frac{1}{2}\left(p^a p_a + m^2\right),$$

Free relativistic particle example

The intrinsic Hamiltonian becomes $H = \frac{1}{2}\left(p^a p_a + m^2\right)$.

Each reduced phase space comes equipped with a Hamiltonian flow.

I can now choose the proper time as the intrinsic evolution parameter by making the gauge choice $\theta = T$ and eliminating its momentum conjugate by solving for P. The result is that the simplectic form becomes

$$dS = \left[-\frac{1}{2m} \left(p^a p_a + m^2 \right) + \ln(\theta) \right] d\theta + p_a dq^a .$$

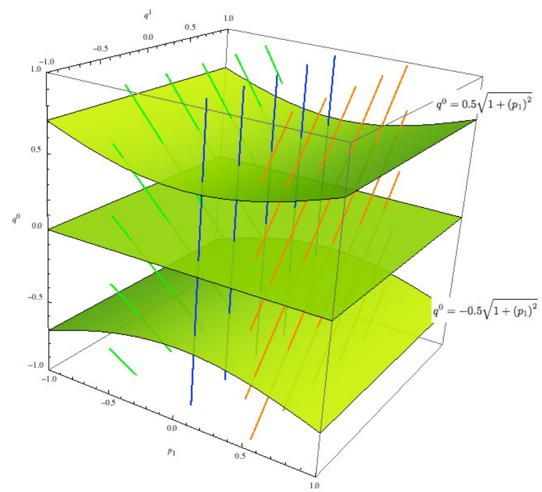

Figure: Proper time slicing in one spatial dimension of free particle gauge orbits, where the proper time values are -0.5, 0, and .5. The particle mass is taken to be one.

An intrinsic H-J approach to GR

The symplectic one form is

$$dS_{GR} = \int d^3x \left(\tilde{p}^{ab} dg_{ab} + \tilde{P}_\mu dN^\mu \right) - dt \int d^3x \left(N^\mu \mathcal{H}_\mu + \lambda^\mu P_\mu \right)$$

Untertake a canonical change of variables to intrinsic coordinates X^μ and their canonical conjugates. This will leave four independent phase space variables g_A and p^B, such that the non-vanishing contribution to the one form becomes

$$\int d^3\theta \left(\pi_\mu d\theta^\mu + p^A dg_A + \frac{\delta F}{\delta g_{ab}} dg_{ab} + \frac{\delta F}{\delta g_A} dg_A + \frac{\delta F}{\delta X^\mu} d\theta^\mu \right)$$

The intrinsic H-J equation

The Hamiltonian becomes a non-trivial functional of the intrinsic coordinates θ^μ, the two independent components of the metric g_A and the two conjugate fields π^B. The corresponding Hamilton-Jacobi equation will yield a complete solution $S[g_A(\theta^a), \theta^\mu; \Pi^B(\theta^b)]$.

Critique of Rovelli's Hamilton-Jacobi proposal

[Rovelli, 2004] has proposed a Hamilton-Jacobi approach to general relativity in which he also employed the free particle as an example. His conclusion was that one could work with entire solution trajectories which in the particle case are fixed by initial data. The example I have exhibited shows that this data is not sufficient to identify physically distinguishable solutions. Each choice of intrinsic time actually yields a different Hamiltonian. And in the generic case in general relativity this Hamiltonian will actually possess its characteristic intrinsic time and space dependence.

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