Léon Rosenfeld's general theory of constrained Hamiltonian dynamics

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References

Léon Rosenfeld, On the Quantization of Wave Fields, *European Journal of Physics H*, (2017), translation by D. Salisbury and K. Sundermeyer of Zur Quantelung der Wellenfelder, *Annalen der Physik*, **5**, 113 (1930)

Léon Rosenfeld's general theory of constrained Hamiltonian dynamics, *European Journal of Physics H*, (with K. Sundermeyer, 2017)

Rosenfeld education and early professional career

- Born 1904 in Charleroi, Belgium
- Completed graduate studies in Paris under the supervision of Louis de Broglie and Théophile de Donder, 1926
- Assistant to Max Born in Göttingen, 1928
- Sought research fellowship, with Einstein's support, to work with Einstein "on the relations between quantum mechanics and relativity"
- Fellowship not awarded and Rosenfeld was invited by Pauli to Zurich, 1929

ROSENFELD'S ACHIEVEMENTS

(1) Primary constraints from Lagrangian symmetries

Free electromagnetic field example. The flat space free electromagnetic field Lagrangian is

$$\mathcal{L}_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

Symmetry transformation

$$\delta A_\mu = \xi_{,\mu}$$

There is a primary constraint expressing the vanishing of the coefficient of $\xi_{,00}$. It is $\frac{\partial \mathcal{L}_{em}}{\partial \dot{A}_0} = p^0 = 0$.

(2) Gauge symmetries always involve singular Lagrangians

The Hessian matrix $\frac{\partial^2 \mathcal{L}_{em}}{\partial \dot{A}_\alpha \partial \dot{A}_0} = \frac{\partial p^0}{\partial \dot{A}_\alpha} = 0$ is singular.

This is a consequence of the invariance of the Lagrangian under the gauge transformation.

(3) Primary constraints lead to Hamiltonians with arbitrary functions

The canonical Hamiltonian is

$$\mathcal{H} = p^{\alpha}\dot{A}_{\alpha} - \mathcal{L}_{em}(A_{\mu,a}, \dot{A}_b(p^c, V_{,d})] = \frac{1}{2}\left(p^a p^a + B_b B_b\right) + p^a A_{0,a} + \lambda p^0,$$

where λ is an arbitrary spacetime function and $B_a = \epsilon_{abc} A_{b,c}$ is the magnetic field.

(4) Vanishing Noether charges expressed as phase space expression

The symmetry identity can be conveniently rewritten in terms of the Euler-Lagrange equations,

$$-\xi_{,\mu}\frac{\partial}{\partial x^{\nu}}\frac{\partial \mathcal{L}_{em}}{\partial A_{\mu,\nu}} + \left(\frac{\partial}{\partial x^{\nu}}\frac{\partial \mathcal{L}_{em}}{\partial A_{\mu,\nu}}\xi_{,\mu}\right)_{,\nu} \equiv 0.$$

We deduce that when the Euler-Lagrange equations are satisfied we have a conserved charge

$$\overline{\mathcal{M}}_{em} =: \int d^3x \left(p^0\dot{\xi} - p^a_{,a}\xi\right),$$

(5) Phase space charges generate correct symmetry variations

The constraint $\overline{\mathcal{M}}_{em}$ generates the infinitesimal symmetry transformations

$$\delta A_\mu = \left\{\delta A_\mu, \overline{\mathcal{M}}_{em}\right\} = \xi_{,\mu},$$

and $\delta p^a = 0$.

(6) Preservation of primary constraints leads to higher order constraints

The achievement (4) may be understood as a derivation of a higher order(secondary) constraint in the sense that if we write $\overline{\mathcal{M}} = \int d^3x \left(p^0 \dot{\xi} + \mathcal{N} \xi \right)$, then we have $\frac{d}{dt}\overline{\mathcal{M}} = \int d^3x \left(\dot{p}^0 \dot{x}i + p^0 \ddot{\xi} + \dot{\mathcal{N}} \xi + \mathcal{N} \dot{\xi} \right) = 0$. The vanishing of the coefficient of $\dot{\xi}$ then yields $\dot{p}^0 = -\mathcal{N} = 0$.

(7) How to construct the Hamiltonian for general relativity

As an example of a generally covariant model we consider the parameterized free relativistic particle. Let $x^\mu(\theta)$ represent the particle spacetime trajectory parameterized by θ. Under a reparameterization $\theta' = f(\theta)$, where f is an arbitrary positive definite function, x^μ transforms as a scalar,

$$x'^\mu(\theta') = x^\mu(\theta).$$

We introduce an auxiliary variable $N(\theta)$ and we assume that it transforms as a scalar density of weight one,

$$N'(\theta') = N(\theta) \left| \frac{d\theta}{d\theta'} \right|^1 .$$

Then the particle Lagrangian takes the form

$$L_p(\dot{x}^\mu, N) = \frac{1}{2N}\frac{dx^\mu}{d\theta}\frac{dx_\mu}{d\theta} - \frac{m^2 N}{2},$$

where $\dot{x}^\mu := \frac{dx^\mu}{d\theta}$.

The conserved charge associated with the free relativistic particle is

$$M_p = p_\mu \delta x^\mu + p_N \delta N - p_\mu \dot{x}^\mu \xi - p_N \dot{N} \xi + L_p \xi$$

$$= -p_N N \dot{\xi} - \frac{N\xi}{2}\left(p^2 + m^2\right) - \lambda p_N \xi,$$

where λ is an arbitrary spacetime function

This does generate the correct infinitesimal symmetry transformations.

It does not generate the correct finite transformations.

(7) The general relativistic Hamiltonian

$$\mathcal{H}_g = \mathcal{H}_g^c + \lambda_I \mathcal{F}^I + \lambda^{[IJ]} \mathcal{F}'_{[IJ]}$$

with

$$\mathcal{H}_g^c = \frac{1}{2} S^{(ab)} M_{(ab)(cd)} S^{(cd)} - N^{(ab)} M_{(ab)(cd)} S^{(cd)}$$

$$+ \frac{1}{2} N^{(ab)} M_{(ab)(cd)} N^{(cd)} - \mathcal{A}$$

where

$$\mathcal{A} = \frac{1}{8\kappa} (-g)^{1/2} \left(4 E_M^{[\alpha} g^{a][b} E_N^{\rho]} - 2 E_N^{[a} g^{\alpha][\rho} E_M^{b]} - \eta_{MN} g^{\rho[\alpha} g^{a]b} \right) e^M_{\alpha,a} e^N_{\rho,b}$$

is the velocity-independent term in $\mathcal{L}_g$.

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