Semi-classical Quantum Gravity

A New Perspective

Donald Salisbury[+] and Jürgen Renn[*]

Austin College, Texas[+]
Max Planck Institute for the History of Science, Berlin[+*]

Institute for the History of Natural Sciences
Beijing, September 7, 2012

Overview

Introduction

Origins of the Wheeler-DeWitt Equation

The Einstein-Hamilton-Jacobi equation of Peres, 1962

IL NUOVO CIMENTO   VOL. XXVI, N. 1   1° Ottobre 1962

On Cauchy's Problem in General Relativity – II.

A. PERES (*) (**)

Palmer Physical Laboratory, Princeton University - Princeton, N. J.

Peres replaces the canonical momenta p^{ab} in the Hamiltonian constraint

$$\mathcal{H} = -\sqrt{g}\,{}^3R + \frac{1}{\sqrt{g}}\left(p^{ab}p_{ab} - \frac{1}{2}p^2\right) = 0,$$

with a functional derivative of a Hamilton principal function S with respect to the spatial metric field g_{ab}

Origins of the Wheeler-DeWitt Equation

The Einstein-Hamilton-Jacobi equation:

$$-g\,{}^{3}R + g_{ab}g_{cd}\left(\frac{\delta S}{\delta g_{ac}}\frac{\delta S}{\delta g_{bd}} - \frac{1}{2}\frac{\delta S}{\delta g_{ab}}\frac{\delta S}{\delta g_{cd}}\right) = 0.$$

Origins of the Wheeler-DeWitt Equation according to DeWitt

From DeWitt's paper "The Quantum and Gravity: The Wheeler-DeWitt equation", *Recent Developments in Theoretical and Experimental General Relativity, Gravitation, and Relativistic Field Theories*, 1999:

"John Wheeler, the perpetuum mobile of physics, called me one day in the early sixties. I was then at the University of North Carolina in Chapel Hill, and he told me that he would be at the Raleigh-Durham airport for two hours between planes. He asked if I could meet with him there and spend a while talking quantum gravity. John was pestering everybody at the time with the question: What are the properties of the quantum mechanical state functional Psi and what is its domain? He had fixed in his mind that the domain must be the space of 3-geometries, and he was seeking a dynamical law for Psi."

Origins of the Wheeler-DeWitt Equation according to DeWitt

"I had recently read a paper by Asher Peres which cast Einstein's theory into Hamilton-Jacobi form, the Hamilton-Jacobi function being a functional of 3-geometries. It was not difficult to follow the path already blazed by Schrödinger, and write down a corresponding wave equation. This I showed to Wheeler, as well as an inner product based on the Wronskian for the functional differential wave operator. Wheeler got tremendoulsy excited at this and began to lecture about it on every occasion."

Origins of the Wheeler-DeWitt Equation according to DeWitt

"I wrote a paper on it in 1965, which didn't get published until 1967 because my Air Force grant was terminated and the Physical Review in those days was holding up publication of papers whose authors couldn't pay the page charges. My heart wasn't really in it because, using a new kind of bracket discovered by Peierls, I had found that I could completely dispense with the cumbersome paraphernalia of constrained hamiltonian systems and build a manifestly gauge covariant quantum theory ab initio. But I thought I should at least point out a number of intriguing features of the functional differential equation, to which no one had yet begun to devote much attention: [...] The fact that the wave functional is a wave function of the universe and therefore cannot be understood except within the framework of a many-worlds view of quantum mechanics [...] In the long run one has no option but

The Wheeler-DeWitt equation

Wheeler-DeWitt equation:

$$-g\,{}^3R + g_{ab}g_{cd}\left(\frac{\delta S}{\delta g_{ac}}\frac{\delta S}{\delta g_{bd}} - \frac{1}{2}\frac{\delta S}{\delta g_{ab}}\frac{\delta S}{\delta g_{cd}}\right) = 0.$$

Supplemented by a Hamilton-Jacobi form of the vector constraint $\mathcal{H}_a = 0$,

$$\left[\frac{\delta S}{\delta g_{ab}}\right]_{|b} = 0,$$

where "|" signifies the three-dimensional covariant derivative. The quantum wave function is taken to be a linear superposition of states

$$\Psi\left[g_{ab}\right] \propto e^{iS/\hbar}.$$

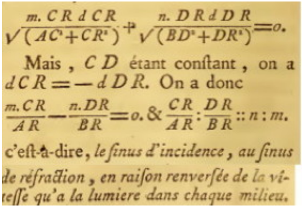
DeWitt's attitude toward the Wheeler-DeWitt equation

Quoted from the abstract of DeWitt's 1999 paper:

"This equation should be confined to the dustbin of history for the following reasons: 1) By focussing on time sclices it violates the very spirit of relativity. 2) Scores of man-yeas have been wasted by researchers trying to extract from it a natural time parameter. 3) Since good path integral techniques exist for basing Quantum Theory on gauge invariant observables only, it seems a pity to drag in the paraphernalia of constrained hamiltonian systems. 4) In the case of mini-superspace models, gauge invariant transition amplitudes defined by the path integral do not satisfy any local differential equation; they satisfy the Wheeler-DeWitt equation only approximately."

Maupertuis' action

Maupertuis believed that light rays followed the path that minimizes the "action" $\int v\, ds$, where v is the speed and ds is the increment of distance.

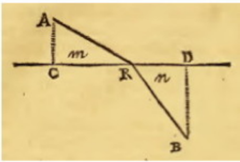


$$\frac{m.CR\,dCR}{\sqrt{(AC^2+CR^2)}} + \frac{n.DR\,dDR}{\sqrt{(BD^2+DR^2)}} = o.$$

Mais, CD étant conſtant, on a $dCR = -dDR$. On a donc

$$\frac{m.CR}{AR} - \frac{n.DR}{BR} = o. \ \& \ \frac{CR}{AR} : \frac{DR}{BR} :: n : m.$$

c'eſt-à-dire, *le ſinus d'incidence, au ſinus de réfraction, en raiſon renverſée de la viteſſe qu'a la lumiere dans chaque milieu.*



Maupertuis' geometrical derivation of the law of refraction from "Accord de différentes loix de la nature qui avoient jusquici paru incompatibles", *Mémoires of the Royal Academy of Science*, 1744

Hamilton's optical mechanical analogy

In a series of papers written in the 1830's, culminating in "Third supplement to an essay on the theory of systems of rays," *Transactions of the Royal Irish Academy*, 1837., Hamilton demonstrated that optical rays minimized an action $\int_{\vec{x}_0}^{\vec{x}_1} n(\vec{x})ds$, where $n(\vec{x})$ is the index of refraction. He choose the distance s as a ray parameter, so the action became

$$W = \int_{s_0}^{s_1} n(\vec{x}) \left(\vec{x}' \cdot \vec{x}'\right)^{1/2} ds =: \int_{s_0}^{s_1} L(\vec{x}, \vec{x}')ds,$$

where $\vec{x}' := \frac{d\vec{x}}{ds}$ and it satisfies the condition $\vec{x}' \cdot \vec{x}' = 1$.

Hamilton's characteristic function

Furthermore Hamilton showed that all relevant information is contained in W when it is evaluated along the actual rays. He called this integral the characteristic function, and he obtained a single partial differential equation that was satisfied by this function.

A remarkable historical fact: Hamilton's original dynamical model was a constrained system. Yet the theory of constrained Hamiltonian dynamical systems was not fully elaborated until the 20th century.

Hamilton's optical mechanical analogy was his observation that both light rays and particle paths could be described by a similar formalism.

The Hamilton-Jacobi equation

Hamilton and Jacobi extended the formalism to a time-dependent action that satisfied Hamilton's principle of least action. They were able to show that when evaluated on solution spacetime particle trajectories, when both the spatial endpoints and the time of arrival were varied, the resulting variation of the action satisfied

$$dS = p_a dx^a - H(x, p)dt,$$

where x^a is the particle position, p_a is the momentum, dt is the time increment, and H is the Hamiltonian. As a consequence, the action S, the Hamilton principal function, satisfies the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(x^a, \frac{\partial S}{\partial x^b}\right) = 0.$$

We will call this the "proper" Hamilton-Jacobi equation.

Jacobi's recovery of solution trajectories

Jacobi showed in 1837 that the full set of classical solutions can be obtained from a complete solution of the principal function. A complete solution $S(x, \alpha)$ is characterized by as many constants α as there are configuration variables x.

ÜBER DIE REDUCTION DER INTEGRATION DER PARTIELLEN DIFFERENTIALGLEICHUNGEN ERSTER ORDNUNG ZWISCHEN IRGEND EINER ZAHL VARIABELN AUF DIE INTEGRATION EINES EINZIGEN SYSTEMES GEWÖHNLICHER DIFFERENTIALGLEICHUNGEN

VON

PROFESSOR C. G. J. JACOBI

ZU KÖNIGSBERG IN PREUSSEN.

Jacobi's 1837 article.

Hamilton-Jacobi approach to quantum mechanics

Schrödinger showed in 1926 that the quantum wave function $\Psi(x,t) = e^{iS/\hbar}$ satisfies to order $1/\hbar^2$ the Schrödinger wave equation

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right)\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

when the principal function $S(x,\alpha)$ satisfies the Hamilton-Jacobi equation.

1926. № 6.

ANNALEN DER PHYSIK.

VIERTE FOLGE. BAND 79.

1. *Quantisierung als Eigenwertproblem;*
von E. Schrödinger.

(Zweite Mitteilung.)[1]

Schrödinger's Second Communication of 1926.

The semi-classical limit

Furthermore, it is possible to show, again based on the satisfaction of the Hamilton-Jacobi equation, that when a suitable superposition over α is undertaken, a wave-packet results that moves along the correct classical trajectory. This is what we call the semi-classical limit.

Brief history of constrained Hamiltonian dynamics

Constrained Hamiltonian dynamics deals with systems that exhibit a local gauge symmetry. Einstein's general relativity is such a theory. Einstein's field equations are covariant under arbitrary changes in the coordinate time and position.

This aspect of general relativity was first addressed in Zurich in 1930, by Léon Rosenfeld, at the request of Wolfgang Pauli. The resulting opus was years ahead of its time, and went largely ignored, even by Rosenfeld himself.

Rosenfeld’s 1930 article

Zur Quantelung der Wellenfelder
Von L. Rosenfeld

Rosenfeld’s *Annalen der Physik* article from 1930.

You can download my translation and commentary as Max Planck preprint 381 at
http://www.mpiwg-berlin.mpg.de/en/resources/preprints.html

T. S. Chang

T. S. Chang, a student of Dirac's in Cambridge in the 1930's, independently rediscovered some of Rosenfeld's results, considered the implications of the formalism for quantum theory, and produced a new procedure for obtaining secondary constraints.

See "Tsung-Sui Chang's Contribution to the Quantization of Constrained Hamiltonian Systems" by Yin Xiaodong, Zhu Zhongyuan and myself, forthcoming.

Bergmann and Dirac

The history of constrained Hamiltonian dynamics has until recently thought to have begun with the work of Bergmann and Dirac.