The Bel–Robinson tensor as an irreducible piece of the Bel tensor

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Motivation & Outline

Bel–Robinson tensor $\tilde{\mathrm{B}}$ is related to the superenergy of the gravitational field.
Bel tensor B is the generalization of Bel–Robinson tensor to non-vacuum spacetimes.

Bel–Robinson tensor in spacetimes of Petrov type D

- cubic invariant in curvature similar to EM: $\tilde{\mathrm{B}}_{\mu\nu\rho\sigma}\tilde{\mathrm{B}}^{\mu\nu\rho\sigma} = 4 \times 12^2 \times \left(\mathbb{E}^2 + \mathbb{B}^2\right)^2$
- principal null directions given by char. surf: $\tilde{\mathrm{B}}_{\mu\nu\rho\sigma}\ell^\mu\ell^\nu\ell^\rho\ell^\sigma = 0$

Bel–Robinson tensor as an energy-momentum-like tensor

- energy-momentum in EM: $\Sigma_\mu := \frac{1}{2}\left[\mathrm{F} \wedge (\mathrm{e}_\mu \lrcorner \star\mathrm{F}) - (\star\mathrm{F}) \wedge \mathrm{e}_\mu \lrcorner \mathrm{F}\right]$
- Bel–Robinson as a 3-form: $\Sigma_{\nu\rho\sigma} := \frac{1}{2}\left[\mathrm{C}_{\rho\alpha} \wedge (\mathrm{e}_\nu \lrcorner \star\mathrm{C}^\alpha{}_\sigma) - (\star\mathrm{C}_{\rho\alpha}) \wedge \mathrm{e}_\nu \lrcorner \mathrm{C}^\alpha{}_\sigma\right]$

"Vacuum" depends on the gravitational theory under consideration. Are there a more general tensors with the same algebraic (!) properties as the Bel–Robinson tensor?

- irreducible decomposition of Bel tensor using Young tableaux; Bel trace tensor

$\rightarrow$ find the most general **algebraic Bel–Robinson tensor**

Warmup: the Riemann tensor and its irreducible decomposition

Symmetries of the Riemann tensor:			#
▪ double 2-form:	$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}$	(alg. curv. tensor)	36
▪ Bianchi identity	$R^{\mu}{}_{[\nu\rho\sigma]} = 0$	(if torsion vanishes)	16
▪ implications:	$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu},\; R_{[\mu\nu\rho\sigma]} = 0$	(if torsion vanishes)	15 + 1

Warmup: the Riemann tensor and its irreducible decomposition

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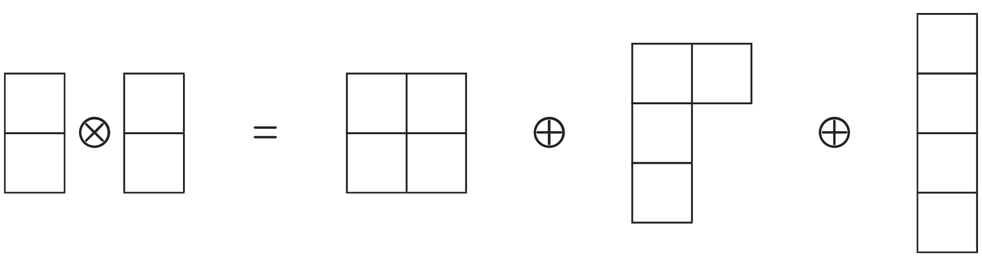
Irreducible decomposition in terms of Young tableaux:



$$R_{\mu\nu\rho\sigma} = \underbrace{\frac{1}{2}\left(R_{\mu\nu\rho\sigma} + R_{\rho\sigma\mu\nu}\right)}_{\text{Weyl, rictf, scalar}} + \underbrace{\left[\frac{1}{2}\left(R_{\mu\nu\rho\sigma} - R_{\rho\sigma\mu\nu}\right) - R_{[\mu\nu\rho\sigma]}\right]}_{\text{paircom, ricanti}} + \underbrace{R_{[\mu\nu\rho\sigma]}}_{\text{pscalar}}$$

$\to$ using the metric $g_{\mu\nu}$, an even finer decomposition is possible by subtracting traces.

Warmup: the Riemann tensor and its irreducible decomposition

Explicit form:

$$
\begin{aligned}
{}^{(2)}\mathsf{R}_{\mu\nu\rho\sigma} &:= \frac{1}{2}\left(\mathsf{R}_{\mu\nu\rho\sigma} - \mathsf{R}_{\rho\sigma\mu\nu}\right) - {}^{(5)}\mathsf{R}_{\mu\nu\rho\sigma}\,,\\
{}^{(3)}\mathsf{R}_{\mu\nu\rho\sigma} &:= -\frac{1}{12}\chi\,\eta_{\mu\nu\rho\sigma}\,,\\
{}^{(4)}\mathsf{R}_{\mu\nu\rho\sigma} &:= \frac{1}{2}\Big(g_{\sigma\nu}\cancel{\mathrm{Ric}}_{(\mu\rho)} - g_{\sigma\mu}\cancel{\mathrm{Ric}}_{(\nu\rho)} + g_{\rho\mu}\cancel{\mathrm{Ric}}_{(\nu\sigma)} - g_{\rho\nu}\cancel{\mathrm{Ric}}_{(\mu\sigma)}\Big)\,,\\
{}^{(5)}\mathsf{R}_{\mu\nu\rho\sigma} &:= \frac{1}{2}\Big(g_{\sigma\nu}\mathrm{Ric}_{[\mu\rho]} - g_{\sigma\mu}\mathrm{Ric}_{[\nu\rho]} + g_{\rho\mu}\mathrm{Ric}_{[\nu\sigma]} - g_{\rho\nu}\mathrm{Ric}_{[\mu\sigma]}\Big)\,,\\
{}^{(6)}\mathsf{R}_{\mu\nu\rho\sigma} &:= \frac{1}{12}\mathsf{R}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)\,,\\
{}^{(1)}\mathsf{R}_{\mu\nu\rho\sigma} &:= \mathsf{R}_{\mu\nu\rho\sigma} - \bigoplus_{\mathsf{I}=2}^{6} {}^{(\mathsf{I})}\mathsf{R}_{\mu\nu\rho\sigma}\,.
\end{aligned}
$$

This refined decomposition is derived entirely from the Ricci tensor $\mathsf{Ric}_{\mu\nu} := \mathsf{R}^{\alpha}{}_{\mu\alpha\nu}$.

Irreducible decomposition of the Bel tensor

Formal definition of the Bel tensor in terms of tensor duals:

$$B_{\mu\nu\rho\sigma} := \frac{1}{2}\Big[R_{\mu\alpha\beta\rho}R_{\nu}{}^{\alpha\beta}{}_{\sigma} \;+\; (*R*)_{\mu\alpha\beta\rho}(*R*)_{\nu}{}^{\alpha\beta}{}_{\sigma}$$
$$+ (*R)_{\mu\alpha\beta\rho}(*R)_{\nu}{}^{\alpha\beta}{}_{\sigma} \;+\; (R*)_{\mu\alpha\beta\rho}(R*)_{\nu}{}^{\alpha\beta}{}_{\sigma}\Big]$$

The Bel tensor has the symmetries $B_{[\mu\nu]\rho\sigma} = B_{\mu\nu[\rho\sigma]} = 0, \quad B^{\alpha}{}_{\alpha\rho\sigma} = 0, \quad B_{\mu\nu}{}^{\alpha}{}_{\alpha} = 0\,.$

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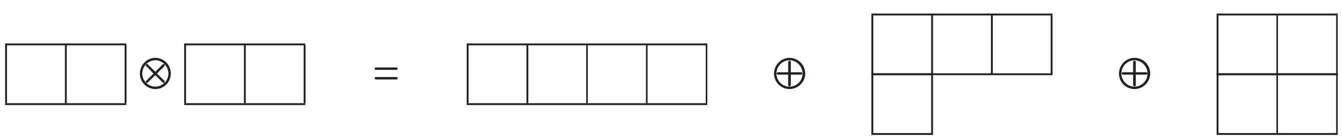
The Bel tensor has the symmetries $B_{[\mu\nu]\rho\sigma} = B_{\mu\nu[\rho\sigma]} = 0, \quad B^{\alpha}{}_{\alpha\rho\sigma} = 0, \quad B_{\mu\nu}{}^{\alpha}{}_{\alpha} = 0\,.$

The irreducible decomposition in terms of Young tableaux is



$$B_{\mu\nu\rho\sigma} \;=\; B_{(\mu\nu\rho\sigma)} + \frac{1}{2}\left(B_{\mu\nu\rho\sigma} - B_{\rho\sigma\mu\nu}\right) \\ + \frac{1}{6}\Big[2\left(B_{\mu\nu\rho\sigma} + B_{\rho\sigma\mu\nu}\right) - \left(B_{\mu\rho\nu\sigma} + B_{\nu\sigma\mu\rho}\right) - \left(B_{\mu\sigma\nu\rho} + B_{\nu\rho\mu\sigma}\right)\Big]$$

Obtain finer decomposition by introducing the Bel trace tensor $B_{\mu\nu} := B^{\alpha}{}_{\mu\alpha\nu}\,.$

Irreducible decomposition of the Bel tensor

Explicit form:

$$^{(1b)}B_{\mu\nu\rho\sigma} := \frac{1}{12}\left(g_{\mu\nu}\not{B}_{\rho\sigma} + g_{\rho\sigma}\not{B}_{\mu\nu} + g_{\mu\rho}\not{B}_{\nu\sigma} + g_{\nu\sigma}\not{B}_{\mu\rho} + g_{\mu\sigma}\not{B}_{\nu\rho} + g_{\nu\rho}\not{B}_{\mu\sigma}\right),$$

$$^{(1c)}B_{\mu\nu\rho\sigma} := \frac{1}{36}B\left(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right),$$

$$^{(1a)}B_{\mu\nu\rho\sigma} := {}^{[1]}B_{\mu\nu\rho\sigma} - {}^{(1b)}B_{\mu\nu\rho\sigma} - {}^{(1c)}B_{\mu\nu\rho\sigma},$$

$$^{(2b)}B_{\mu\nu\rho\sigma} := \frac{1}{6}\left(g_{\mu\rho}B_{[\nu\sigma]} + g_{\nu\rho}B_{[\mu\sigma]} + g_{\mu\sigma}B_{[\nu\rho]} + g_{\nu\sigma}B_{[\mu\rho]}\right),$$

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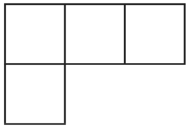
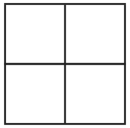


$$^{(3b)}B_{\mu\nu\rho\sigma} := \frac{1}{6}\left(g_{\mu\rho}\not{B}_{\nu\sigma} + g_{\nu\sigma}\not{B}_{\mu\rho} + g_{\mu\sigma}\not{B}_{\nu\rho} + g_{\nu\rho}\not{B}_{\mu\sigma} - 2g_{\mu\nu}\not{B}_{\rho\sigma} - 2g_{\rho\sigma}\not{B}_{\mu\nu}\right),$$

$$^{(3c)}B_{\mu\nu\rho\sigma} := \frac{1}{36}B\left(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho} - 2g_{\mu\nu}g_{\rho\sigma}\right),$$

$$^{(3a)}B_{\mu\nu\rho\sigma} := {}^{[3]}B_{\mu\nu\rho\sigma} - {}^{(3b)}B_{\mu\nu\rho\sigma} - {}^{(3c)}B_{\mu\nu\rho\sigma}.$$



The most general algebraic Bel–Robinson tensor

The following is our final result:

$$
\begin{aligned}
{}^{(1a)}\mathrm{B}_{\mu\nu\rho\sigma} &:= {}^{[1]}\mathrm{B}_{\mu\nu\rho\sigma} - {}^{(1b)}\mathrm{B}_{\mu\nu\rho\sigma} - {}^{(1c)}\mathrm{B}_{\mu\nu\rho\sigma}\,,\\
{}^{(1b)}\mathrm{B}_{\mu\nu\rho\sigma} &:= \frac{1}{12}\left(g_{\mu\nu}\not{\mathrm{B}}_{\rho\sigma} + g_{\rho\sigma}\not{\mathrm{B}}_{\mu\nu} + g_{\mu\rho}\not{\mathrm{B}}_{\nu\sigma} + g_{\nu\sigma}\not{\mathrm{B}}_{\mu\rho} + g_{\mu\sigma}\not{\mathrm{B}}_{\nu\rho} + g_{\nu\rho}\not{\mathrm{B}}_{\mu\sigma}\right),\\
{}^{(1c)}\mathrm{B}_{\mu\nu\rho\sigma} &:= \frac{1}{36}\mathrm{B}\left(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{B}_{\mu\nu} &:= \mathrm{B}^{\alpha}{}_{\mu\alpha\nu} =: \not{\mathrm{B}}_{\mu\nu} \oplus \mathrm{B}_{[\mu\nu]} \oplus \frac{1}{4}\mathrm{B}g_{\mu\nu}\,,\\
\not{\mathrm{B}}_{\mu\nu} &= {}^{(2)}\mathrm{R}_{\mu\alpha\beta\gamma}\,{}^{(2)}\mathrm{R}^{\alpha\beta\gamma}{}_{\nu} - g^{\alpha\beta}\left(2\mathrm{Ric}_{[\mu\alpha]}\mathrm{Ric}_{[\nu\beta]} + \cancel{\mathrm{Ric}}_{\mu\alpha}\cancel{\mathrm{Ric}}_{\nu\beta}\right)\\
&\quad + \frac{1}{4}g_{\mu\nu}\left(2\mathrm{Ric}_{[\alpha\beta]}\mathrm{Ric}^{[\alpha\beta]} + \cancel{\mathrm{Ric}}_{\alpha\beta}\cancel{\mathrm{Ric}}^{\alpha\beta}\right),\\
\mathrm{B}_{[\mu\nu]} &= \frac{1}{2}\left(\mathrm{R}\mathrm{Ric}_{[\mu\nu]} + \frac{1}{2}\chi\eta_{\mu\nu\alpha\beta}\mathrm{Ric}^{[\alpha\beta]}\right),\\
{}^{(3)}\mathrm{B}_{\mu\nu} &= \frac{1}{4}\left(-\frac{1}{2}\,{}^{(2)}\mathrm{R}_{\alpha\beta\gamma\delta}\,{}^{(2)}\mathrm{R}^{\alpha\beta\gamma\delta} + \cancel{\mathrm{Ric}}_{\alpha\beta}\cancel{\mathrm{Ric}}^{\alpha\beta} + \frac{1}{4}\mathrm{R}^2 + \frac{1}{4}\chi^2\right)g_{\mu\nu}\,.
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The following is our **final result**:

$$^{(1a)}B_{\mu\nu\rho\sigma} := {}^{[1]}B_{\mu\nu\rho\sigma} - {}^{(1b)}B_{\mu\nu\rho\sigma} - {}^{(1c)}B_{\mu\nu\rho\sigma},$$

$$^{(1b)}B_{\mu\nu\rho\sigma} := \frac{1}{12}\left(g_{\mu\nu}\not{B}_{\rho\sigma} + g_{\rho\sigma}\not{B}_{\mu\nu} + g_{\mu\rho}\not{B}_{\nu\sigma} + g_{\nu\sigma}\not{B}_{\mu\rho} + g_{\mu\sigma}\not{B}_{\nu\rho} + g_{\nu\rho}\not{B}_{\mu\sigma}\right),$$

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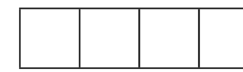

$$B_{\mu\nu} := B^{\alpha}{}_{\mu\alpha\nu} =: \not{B}_{\mu\nu} \oplus B_{[\mu\nu]} \oplus \frac{1}{4}Bg_{\mu\nu},$$

$$\not{B}_{\mu\nu} = {}^{(2)}R_{\mu\alpha\beta\gamma}{}^{(2)}R^{\alpha\beta\gamma}{}_{\nu} - g^{\alpha\beta}\left(2\mathrm{Ric}_{[\mu\alpha]}\mathrm{Ric}_{[\nu\beta]} + \cancel{\mathrm{Ric}}_{\mu\alpha}\cancel{\mathrm{Ric}}_{\nu\beta}\right)$$
$$+ \frac{1}{4}g_{\mu\nu}\left(2\mathrm{Ric}_{[\alpha\beta]}\mathrm{Ric}^{[\alpha\beta]} + \cancel{\mathrm{Ric}}_{\alpha\beta}\cancel{\mathrm{Ric}}^{\alpha\beta}\right),$$

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$$^{(3)}B_{\mu\nu} = \frac{1}{4}\left(-\frac{1}{2}{}^{(2)}R_{\alpha\beta\gamma\delta}{}^{(2)}R^{\alpha\beta\gamma\delta} + \cancel{\mathrm{Ric}}_{\alpha\beta}\cancel{\mathrm{Ric}}^{\alpha\beta} + \frac{1}{4}R^2 + \frac{1}{4}\chi^2\right)g_{\mu\nu}.$$

The most general algebraic Bel–Robinson tensor

The Bel trace tensor lists how different curvature ingredients contribute to traces:

- In General Relativity, $\mathrm{Ric}_{\mu\nu} = 0$ implies $\mathrm{B}_{\mu\nu} = 0$.
- In other theories (different Lagrangian, different geometry with torsion, ...), the vacuum field equations may impose other constraints on the curvature.
- Only the Weyl tensor does not appear in the Bel trace tensor. This is because it is traceless, ${}^{(1)}\mathrm{R}^{\alpha}{}_{\mu\alpha\beta} = 0$, and it also satisfies ${}^{(1)}\mathrm{R}^{\mu}{}_{[\nu\rho\sigma]} = 0$.

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Conclusions

The Bel trace tensor allows us to define a tensor that has the same algebraic properties as the Bel–Robinson tensor. Further work needs to be done:

- Would a spinorial treatment give rise to a deeper algebraic understanding?
- What about differential properties of the algebraic Bel–Robinson tensor?

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Thank you for your attention.