Testing theories of quantum gravity in the early universe

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Table of Contents

Based on:

- Effects of pre-inflation dynamics and universalities of the evolutions of the background and its perturbations in loop quantum cosmology, in preparation (2017).
- Universal features of quantum bounce in loop quantum cosmology, arXiv:1607.06329.
- High-order Primordial Perturbations with Quantum Gravitational Effects, PRD93, 123525 (2016) [arXiv:1604.05739]
- Inflationary spectra with inverse-volume corrections in loop quantum cosmology and their observational constraints from Planck 2015 data, JCAP 03 (2016) 046 [arXiv:1510.03855]
- Scalar and tensor perturbations in loop quantum cosmology: High-order corrections, JCAP 10 (2015) 052 [arXiv:1508.03239]
- Detecting quantum gravitational effects of loop quantum cosmology in the early universe? ApJL 807 (2015) L17 [arXiv:1503.06761]

Based on (Cont.):

- Power spectra and spectral indices of k-inflation: high-order corrections, PRD90 (2014) 103517 [arXiv:1407.8011]
- Quantum effects on power spectra and spectral indices with higher-order corrections,
  PRD90 (2014) 063503 [arXiv:1405.5301]
- Inflationary cosmology with nonlinear dispersion relations,
  PRD89 (2014) 043507 [arXiv:1308.5708]
- Constructing analytical solutions of linear perturbations of inflation with modified dispersion relations,
  IJMPA29 (2014) 1450142 [arXiv:1308.1104]
- Vector and tensor perturbations in Horava-Lifshitz cosmology,
  PRD82 (2010) 124063 (2010) [arXiv:1008.3637]

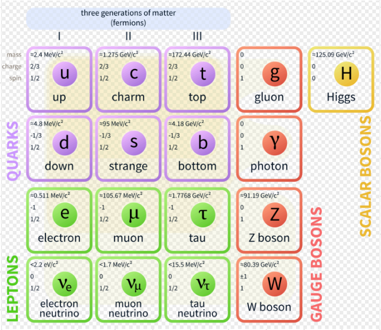
Table of Contents

1.1 The need of Observational Evidences

Four forces in nature:

Electromagnetic, Weak, Strong

Standard Model



Gravitational

???

- String/M-Theory
- Loop Quantum Gravity
- Asymptotic Safety
- noncommutative Gravity
- Causal Dynamical Triangulations
- Hořava-Lifshitz Gravity
- ...

1.1 The need of Observational Evidences (Cont.)

- Gravitational force is weak:

Interaction	Current theory	Mediators	Relative strength[4]	Long-distance behavior	Range (m)[citation needed]
Strong	Quantum chromodynamics (QCD)	gluons	10^{38}	1 (see discussion below)	10^{-15}
Electromagnetic	Quantum electrodynamics (QED)	photons	10^{36}	$\frac{1}{r^2}$	∞
Weak	Electroweak Theory (EWT)	W and Z bosons	10^{25}	$\frac{1}{r}\, e^{-m_{W,Z}\, r}$	10^{-18}
Gravitation	General relativity (GR)	gravitons (hypothetical)	1	$\frac{1}{r^2}$	∞

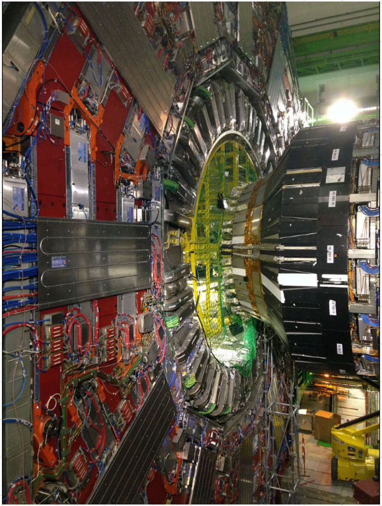
- Effects of Quantum Gravity

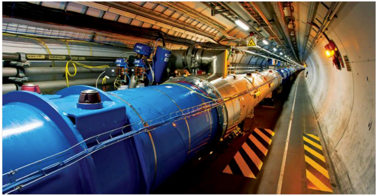
  are expected to become important when:

  - Energy $\simeq 10^{15}$ TeV
  - Size $\simeq 10^{-35}$ m
  - Time $\simeq 10^{-44}$ s

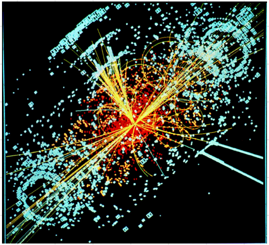
1.1 The need of Observational Evidences (Cont.)

- Energy of LHC: 14 TeV $\left[\ll 10^{15}\text{ TeV}\right]$





The Large Hadron Collider is the world's largest and most powerful particle accelerator (Image: CERN)

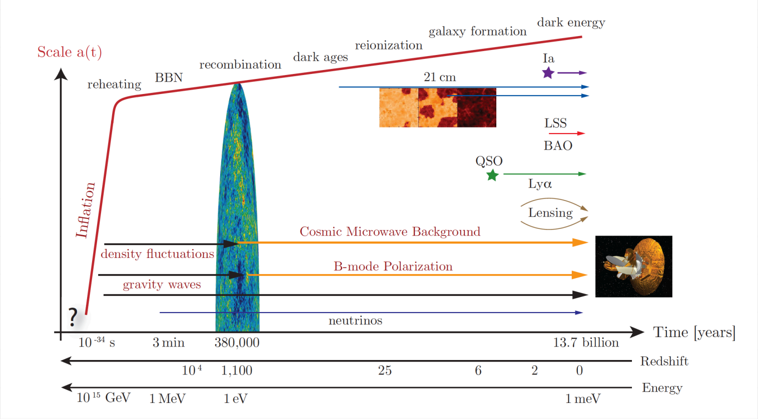


1.1 The need of Observational Evidences (Cont.)

- Energy of Higgs Factory: 100 TeV $[\ll 10^{15}$ TeV$]$

1.1 The need of Observational Evidences (Cont.)

- Energy of Inflation: 10^{13} TeV $[\lesssim 10^{15}$ TeV$]$



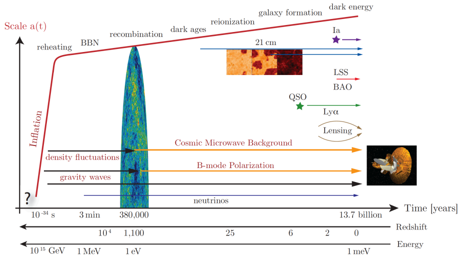
1.2 Inflation is sensitive to QG

- Trans-Planckian Problem [1]:
  - During inflation the wavelengths, related to present observations, were exponentially stretched,

$$e^{N_{\mathrm{inf.}}} = \frac{a_{\mathrm{end}}}{a_i}$$

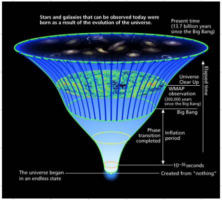
  - To be consistent with observations, we need,

$$N_{\mathrm{inf.}} \geq 60$$



[1] Brandenberger & Martin, CGQ30 (2013) 113001]

1.2 Inflation is sensitive to QG



- Trans-Planckian Problem [2]:
  - If

$$N_{\text{inf.}} > 72,$$

then,

$$\begin{aligned} a_i &= e^{-N_{\text{inf.}}} \cdot a_{\mathrm{end}} = a_0 \cdot \left(\frac{a_{\mathrm{end}}}{a_0}\right) e^{-N_{\text{inf.}}} \\ &< a_0 \cdot e^{-(72+60)} < l_{\mathrm{Pl}} \equiv 10^{-35}\mathrm{m}, \end{aligned}$$

that is, the wavelengths corresponding to present observations, should be smaller than the Planck length at the beginning of the inflation, and quantum gravity becomes important.

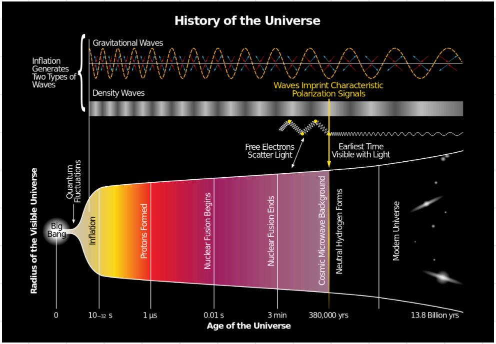
[2] Brandenberger & Martin, CGQ30 (2013) 113001]

1.2 Inflation is sensitive to QG (Cont.)

- Trans-Planckian Problem (Cont.):
  - Then, the assumption adopted in inflationary cosmology that the spacetime is classical becomes questionable, and quantum physics of gravity must be taken into account — the trans-Planckian problem.

1.2 Inflation is sensitive to QG (Cont.)

- Initial singularity problem:
  - General relativity (GR) inevitably leads inflation to an initial singularity [3], with which it is not clear how to impose the initial conditions.



[3] A. Borde and A. Vilenkin, PRL72 (1994) 3305; A. Borde, A. H. Guth, and A. Vilenkin, PRL90 (2003) 151301.

1.2 Inflation is sensitive to QG (Cont.)

- Initial conditions problem:
  - Many inflationary scenarios only work if the fields are initially *very homogeneous and/or start with precise initial positions and velocities*.
  - Any physical understanding of this "fine-tuning" requires a more complete formulation with ever-higher energies, such as string theory.
- ...

1.2 Inflation is sensitive to QG (Cont.)

- Therefore:

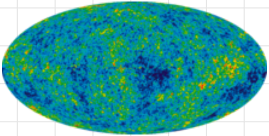
Inflation is very sensitive to Planck-scale physics, and effects of quantum gravity in the early universe are important and need to be taken into account [4].

[4] D. Baumann, TASI Lectures on Inflation, arXiv:0907.5424
C.P. Burgess, M. Cicoli, F. Quevedo, JCAP 1311 (2013) 003
D. Baumann and L. McAllister, Inflation and String Theory (Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2015)
E. Silverstein, TASI lectures on cosmological observables and string theory, arXiv:1606.03640.

1.3 Era of Precision Cosmology

- Inflation is remarkably successful and its predictions are matched to observations with astonishing precision. [5]
- According to the inflation paradigm, the large-scale structure of our universe and CMB all originated from *quantum fluctuations produced during Inflation, which can be decomposed into:*

Scalar, Vector, Tensor



- But, because of the expansion of the universe and particular nature of the fluctuations, vector perturbations did not grow, and observationally can be safely ignored:

$$\text{vector perturbations} \simeq 0$$

[5] Planck Collaboration, arXiv:1507.02704.

1.3 Era of Precision Cosmology(Cont.)

- Scalar and tensor perturbations are described by mode functions $\mu_k(\eta)$,

$$\mu_k'' + \left(\omega_k^2 - \frac{z''}{z}\right)\mu_k = 0, \quad z \equiv \begin{cases} \frac{a\phi'}{\mathcal{H}}, & \text{scalar} \\ a, & \text{tensor} \end{cases} \tag{1}$$

  - ω_k^2: energy of the mode, and in general relativity (GR) is given by,

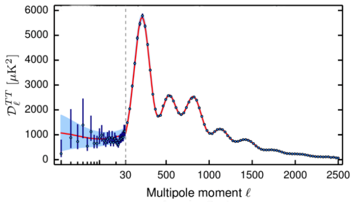
$$\omega_k^2 = k^2$$

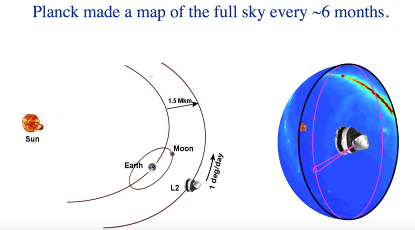
  - k: comoving wavenumber
  - ϕ: the scalar field — the inflaton; and $\phi' \equiv d\phi/d\eta$
  - η: the conform time, $d\eta \equiv dt/a(t)$
  - $a(\eta)$: the expansion factor of the universe; and $\mathcal{H} \equiv a'/a$

1.3 Era of Precision Cosmology(Cont.)

- Power spectra, $\Delta_{\rm i}^2$, are defined as,

$$\Delta_{\rm i}^2 \equiv \frac{{\rm k}^3}{2\pi^2}\left|\frac{\mu_{\rm k}}{\rm z}\right|_{\rm i}^2, \ ({\rm i} = {\rm S}, {\rm T}).$$





(Planck2015, arXiv:arXiv:1502.02114)

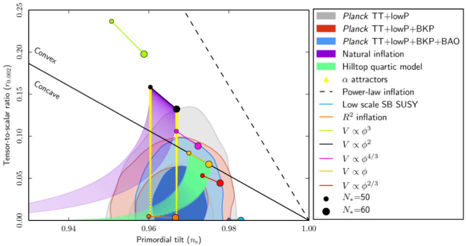
1.3 Era of Precision Cosmology(Cont.)

- Spectral indexes are defined as

$$n_s \equiv 1 + \frac{d \ln \Delta_s^2(k)}{d \ln k}, \quad n_T \equiv \frac{d \ln \Delta_T^2(k)}{d \ln k}.$$

- The ratio r is defined as,

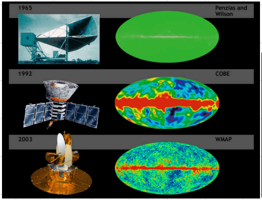
$$r \equiv \frac{\Delta_T^2}{\Delta_S^2}.$$

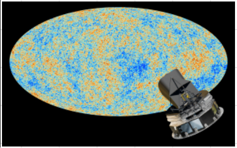


(Planck2015, arXiv:arXiv:1502.02114)

1.3 Era of Precision Cosmology(Cont.)

- Since the first measurement of CMB in 1964 by Penzias and Wilson (PW), there have been a variety of experiments to measure its radiation anisotropies and polarization, such as WMAP, PLanck and BICEP2, with ever increasing precision.





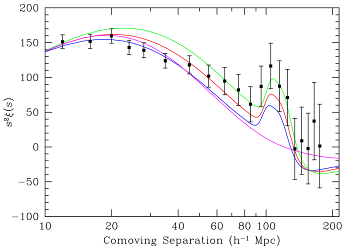
PW, COBE, WMAP | Planck | BICEP2

1.3 Era of Precision Cosmology(Cont.)

- In the coming decade, we anticipate that various new surveys will make even more accurate CMB measurements:
  - Balloon experiments: Balloon-borne Radiometers for Sky Polarisation Observations (BaR-SPoRT); The E and B Experiment (EBEX); ...
  - Ground experiments: Cosmology Large Angular Scale Surveyor (CLASS); Millimeter-Wave Bolometric Interferometer (MBI-B); Qubic; ...
  - Space experiments: Sky Polarization Observatory (SPOrt); ...

1.3 Era of Precision Cosmology(Cont.)

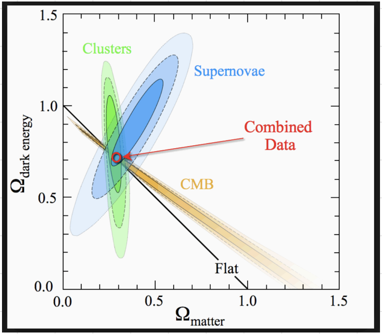
- In addition to CMB measurements, Large-scale structure surveys, measuring the galaxy power spectrum and the position of the baryon acoustic peak, have provided independently valuable information on the evolution of the universe.
- The first measurement of the kind started with the baryon acoustic oscillation (BAO) in the SDSS LRG and 2dF Galaxy surveys [6].



[6] D. J. Eisenstein, et al., ApJ 633 (2005) 560; S. Cole, et al., MNRAS362 (2005) 505.

1.3 Era of Precision Cosmology(Cont.)

- Since then, various large-scale structure surveys have been carried out [7], and provided sharp constraint on the budgets that made of the universe.



[7] Tegmark, M., et al. 2006, Phys. Rev. D, 74, 123507; Kazin, E. A., et al. 2010, ApJ, 710, 1444; Blake, C., Kazin, E., Beutler, F., et al. 2011, MNRAS, 418, 1707.

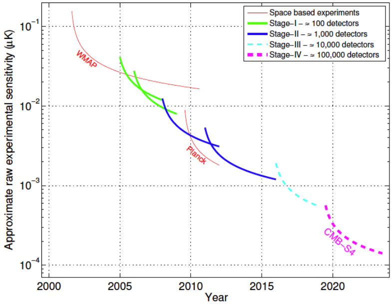
1.3 Era of Precision Cosmology(Cont.)

- Various new surveys will make even more accurate measurements of the galaxy power spectrum:
  - Ground-Based: the Prime Focus Spectrograph, Big BOSS, ...
  - Space-based: Euclid, WFIRST, ...
- Cosmology indeed enters its golden age:

1.3 Era of Precision Cosmology(Cont.)

- In particular, the Stage IV experiments will measure the physical variables $\mathrm{n_s}$ and r with the accuracy[8]:

$$\sigma(\mathrm{n_s}, \mathrm{r}) = 10^{-3} \sim 10^{-4} \qquad (2)$$



[8] K.N. Abazajian et al., "Inflation physics from the cosmic microwave background and large scale structure", Astropart. Phys. 63, 55 (2015) [arXiv:1309.5381]; arXiv:1610.02743.

1.3 Era of Precision Cosmology(Cont.)

- With this level of uncertainty, the Stage IV experiments will make a clear detection ($> 5\sigma$) of tensor modes from any inflationary model with

$$\mathrm{r} \geq 0.01$$

- Note that current measurements from Planck 2015 (Stage II) [Planck Collaboration, arXiv:1502.02114] are,

$$\mathrm{n_s} = 0.968 \pm 0.006,$$
$$\mathrm{r} < 0.11 \ (95\,\%\ \mathrm{CL})$$

1.3 Era of Precision Cosmology(Cont.)

- Gravity Research Foundation:

THE MAIN PURPOSE OF THE GRAVITY RESEARCH FOUNDATION IS TO ENCOURAGE SCIENTIFIC RESEARCH AND TO ARRIVE AT A MORE COMPLETE UNDERSTANDING OF THE PHENOMENON OF GRAVITATION THROUGH ITS ANNUAL AWARDS FOR ESSAYS ON GRAVITATION WITH THE EXPECTATION THAT BENEFICIAL USES WILL ENSUE.

Nicolas Copernicus – *"I am aware that a philosopher's ideas are not subject to the judgment of ordinary persons, because it is his endeavor to seek the truth in all things, to the extent permitted to human reason by God."*

Galileo Galilei – *"In questions of science, the authority of a thousand is not worth the humble reasoning of a single individual."*

Johannes Kepler – *"The diversity of nature is so great, and the treasures hidden in the heavens so rich, precisely in order that the human mind shall never be lacking in fresh nourishment."*

Isaac Newton – *"If I have ever made any valuable discoveries, it has been owing more to patient attention, than to any other talent."*
"This most beautiful system [The Universe] could only proceed from the dominion of an intelligent and powerful Being."

Michael Faraday – *"Nothing is too wonderful to be true if it is consistent with the laws of nature."*

James Clerk Maxwell – *"The only laws of matter are those that our minds must fabricate and the only laws of mind are fabricated for it by matter."*

Albert Einstein – *"All that is valuable in human society depends upon the opportunity for development accorded to the individual."*

- The 2012 First Award:

International Journal of Modern Physics D
Vol. 21, No. 11 (2012) 1241001 (5 pages)

DOI: 10.1142/S0218271812410015

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CAN EFFECTS OF QUANTUM GRAVITY BE OBSERVED IN THE COSMIC MICROWAVE BACKGROUND?*

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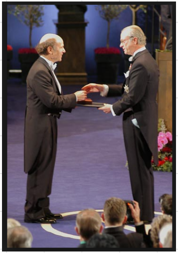
1.3 Era of Precision Cosmology(Cont.)

- The 2014 First Award:

Krauss, Wilczek honored with first prize from Gravity Research Foundation

June 9, 2014

Arizona State University professors Lawrence Krauss and Nobel laureate Frank Wilczek have been named first place winners of the 2014 Awards for Essays from the Gravity Research Foundation, Wellesley Hills, Massachusetts.



Lawrence M. Krauss

Krauss at Ghent University, October 17, 2013

1.3 Era of Precision Cosmology(Cont.)

- The 2014 First Award (Cont.):

International Journal of Modern Physics D
Vol. 23, No. 12 (2014) 1441001 (6 pages)

DOI: 10.1142/S0218271814410016

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From *B*-modes to quantum gravity and unification of forces*

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Table of Contents

2.1 Mathematical Challenges of the Problem

- The problem becomes very complicated after quantum gravitational effects are taken into account.
- Currently known methods, such as WKB and Green functions, are either cannot apply to these cases or produce very large errors [9], far above the accuracy required by current observations [10].

[9] S.E. Joras and G. Marozzi, Phys. Rev. D79 (2009) 023514
A. Ashoorioon, D. Chialva and U. Danielsson, J. Cosmol. Astropart. Phys. 06, 034 (2011).

[10] K.N. Abazajian et al., “Inflation physics from the cosmic microwave background and large scale structure”, Astropart. Phys. 63, 55 (2015) [arXiv:1309.5381]; arXiv:1610.02743.

2.1 Mathematical Challenges of the Problem (Cont.)

PHYSICAL REVIEW D **79,** 023514 (2009)

Trans-Planckian physics from a nonlinear dispersion relation

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(Received 28 August 2008; published 22 January 2009)

We study a particular nonlinear dispersion relation $\omega_p(k_p)$—a series expansion in the physical wave number k_p—for modeling first-order corrections in the equation of motion of a test scalar field in a de Sitter spacetime from trans-Planckian physics in cosmology. Using both a numerical approach and a semianalytical one, we show that the WKB approximation previously adopted in the literature should be used with caution, since it holds only when the comoving wave number $k \gg aH$. We determine the amplitude and behavior of the corrections on the power spectrum for this test field. Furthermore, we consider also a more realistic model of inflation, the power-law model, using only a numerical approach to determine the corrections on the power spectrum.

DOI: 10.1103/PhysRevD.79.023514 PACS numbers: 98.80.Cq, 98.70.Vc

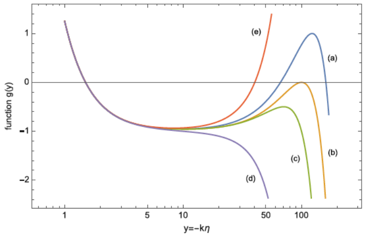
2.1 Mathematical Challenges of the Problem (Cont.)

- The main reason is that the WKB approximation condition $W \ll 1$ is not always satisfied, where

$$W \equiv \left| \frac{1}{\Omega_k^2} \left[\frac{3}{4} \left(\frac{\Omega_k'}{\Omega_k} \right)^2 - \frac{1}{2} \frac{\Omega_k''}{\Omega_k} \right] \right|, \qquad \mu_k'' + \Omega_k^2 \mu_k = 0.$$

- Example:

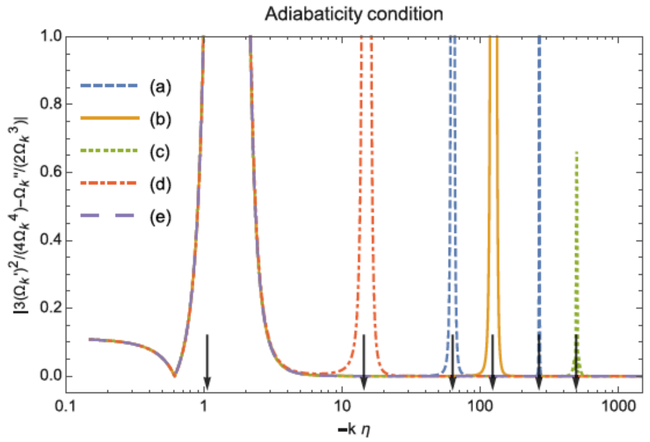
$$\begin{aligned} \Omega_k^2 &= k^2 \left(1 - \hat{b}_1 \frac{k^2}{a^2} + \hat{b}_1 \frac{k^4}{a^4} \right) \\ g &\equiv -\left(\frac{1}{4y^2} + \frac{\Omega_k^2}{k^2} \right) \end{aligned}$$



$\hat{b}_1, \hat{b}_2$: constants

Zhu et al, PRD93 (2016) 123525

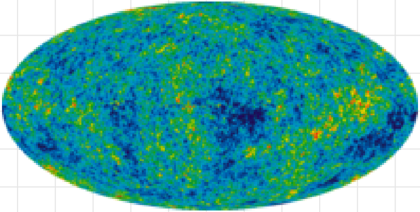
2.1 Mathematical Challenges of the Problem (Cont.)



Zhu et al, PRD93 (2016) 123525

2.1 Mathematical Challenges of the Problem (Cont.)

- Quantum gravitational effects are expected to be small. So, theoretical calculations of physical observables with high accuracy are highly demanded.



2.2 Quantum Gravitational Effects

String/M-Theory:

- As the most promising candidate for a UV-completion of the Standard Model that unifies gauge and gravitational interaction in a consistent quantum theory, String/M theory can provide possibilities for an explicit realization of the inflationary scenario.
- String/M theory usually leads to a non-trivial time-dependent speed of sound for primordial perturbations [11],

$$\omega_k^2 = c_s^2(\eta) k^2, \tag{3}$$

  - $c_s^2(\eta)$: the speed of sound, and could be very close to zero in the far UV regime.

[11] L. McAllister and E. Silverstein, Gen. Rel. Grav. 40, 565 (2007); C. P. Burgess, M. Cicoli, and F. Quevedo, arXiv: 1306.3512.

Loop Quantum Cosmology:

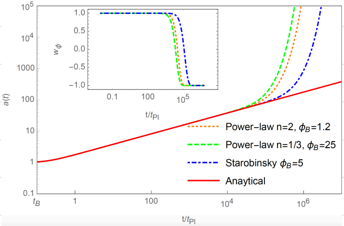
- Offers a natural framework to address the trans-Planckian issue and initial singularity:

$$\mathrm{H}^2 \equiv \left(\frac{\dot{\mathrm{a}}}{\mathrm{a}}\right)^2 = \frac{8\pi \mathrm{G}}{3}\rho\left(1-\frac{\rho}{\rho_{\mathrm{B}}}\right), \quad \rho_{\mathrm{B}} \simeq 0.41 \mathrm{m}_{\mathrm{pl}}^4.$$

- The universe starts to expand at $\rho = \rho_{\mathrm{B}}$ with

$$\mathrm{a}(\mathrm{t}_{\mathrm{B}}) > 0$$

— quantum bounce.



2.2 Quantum Gravitational Effects (Cont.)

- There are mainly two different approaches to study cosmological perturbations in LQC:

  (A) Deformed Algebra

  (B) Dressed Metric

- In the framework of the deformed algebra approach, there are mainly two kinds of quantum corrections [12]:

  A.1) holonomy

  A.2) inverse-volume

[12] A. Barrau and J. Grain, arXiv:1410.1714; M. Bojowald, Rep. Prog. Phys. 78 (2015) 023901; A. Ashtekar and A. Barrau, arXiv:1504.07559.

2.2 Quantum Gravitational Effects (Cont.)

- Due to the holonomy corrections, the dispersion relation in the mode functions is modified to

$$\omega_{\mathrm{k}}^2 = \left(1 - 2\frac{\rho}{\rho_{\mathrm{B}}}\right)\mathrm{k}^2 \tag{4}$$

- Due to the inverse-volume corrections, the dispersion relation in the mode functions is modified to

$$\omega_{\mathrm{k}}^2 = \mathrm{k}^2 \times \begin{cases} 1 + \left[\frac{\sigma\nu_0}{3}\left(\frac{\sigma}{6}+1\right) + \frac{\alpha_0}{2}\left(5 - \frac{\sigma}{3}\right)\right]\delta_{\mathrm{PL}}(\eta), & \texttt{scalar} \\ 1 + 2\alpha_0\delta_{\mathrm{PL}}, & \texttt{tensor} \end{cases} \tag{5}$$

  - $\alpha_0,\ \nu_0,\ \sigma$: encode the specific features of the model
  - $\delta_{\mathrm{PL}}(\eta)$: time-dependent, given by $\delta_{\mathrm{PL}} = (\mathrm{a}_{\mathrm{PL}}/\mathrm{a})^{\sigma} < 1,$ with $\sigma > 0$.

2.2 Quantum Gravitational Effects (Cont.)

- In the framework of *the dressed metric approach* [13], the dispersion relation in the mode functions is modified to

$$\omega_k^2 = k^2 - \frac{a''}{a} + U(\eta) \tag{6}$$

where

$$U(\eta) = \begin{cases} a^2\left(\mathfrak{f}^2 V(\phi) + 2\mathfrak{f} V_{,\phi}(\phi) + V_{,\phi\phi}(\phi)\right), & \texttt{scalar} \\ 0, & \texttt{tensor} \end{cases} \tag{7}$$

$$\mathfrak{f} \equiv \sqrt{24\pi G}\dot{\phi}/\sqrt{\rho}.$$

[13] A. Barrau and J. Grain, arXiv:1410.1714; M. Bojowald, Rep. Prog. Phys. 78 (2015) 023901; A. Ashtekar and A. Barrau, arXiv:1504.07559.

2.2 Quantum Gravitational Effects (Cont.)

Horava-Lifshitz (HL) Gravity:

- To quantize gravity using quantum field theory, in 2009 Horava proposed a theory - HL gravity [14], which is power-counting renormalizable, and has attracted lot of attention since then.
- In this theory, the dispersion relation is modified to [15],

$$\omega_k^2(\eta) = k^2 - b_1 \frac{k^4}{a^2 M_*^2} + b_2 \frac{k^6}{a^4 M_*^4} \tag{8}$$

  - M_*: the energy scale of the HL gravity
  - b_1, b_2: depend on the coupling constants of the HL theory and the type of perturbations, scalar or tensor.

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[14] P. Horava, PRD79 (2009) 084008
AW, Horava Gravity at a Lifshitz Point: A Progress Report, Inter. J. Mod. Phys. D26 (2017) 1730014 [arXiv:1701.06087].

[15] AW and R. Maartens, PRD81 (2010) 024009; AW, PRD82 (2010) 124063.

2.3 Uniform Asymptotic Approximation Method

- Taking the quantum effects into account, either from *string/ M-Theory, or loop Quantum Cosmology, or HL gravity,* or any of other theories, the equation of motion for the mode function μ_k in general can be cast in the form,

$$\frac{d^2\mu_k(y)}{dy^2} = \left[g(y) + q(y)\right]\mu_k(y), \tag{9}$$

$g(y),\ q(y)$: functions of $y[\equiv -k\eta]$, *to be determined by minimizing the errors.*

- For example, in the HL gravity, we have

$$g(y) + q(y) \quad = \quad \frac{\nu^2 - 1/4}{y^2} - 1 + b_1\epsilon_*^2 y^2 - b_2\epsilon_*^4 y^4,$$

with $\epsilon_* \equiv H/M_*, z''/z \equiv (\nu^2(\eta) - 1/4)/\eta^2$.

2.3 Uniform Asymptotic Approximation Method (Cont.)

- The strategy is, following Olvier [16], to use the well-established Liouville transformations to introduce
  - a new variable ξ, instead of y,
  - a new function U, instead of μ_{k},

$$\mathrm{y} \to \xi, \qquad \mu_{\mathrm{k}}(\mathrm{y}) \to \mathrm{U}(\xi),$$

- so that the resulted equation can be solved:
  - analytically order by order in terms of $1/\lambda \ll 1$
  - the corresponding error bounds are minimized

[16] F. W. J. Olver, Asymptotics and Special functions, (AKP Classics, Wellesley, MA 1997).

2.3 Uniform Asymptotic Approximation Method (Cont.)

- The Liouville Transformations are

$$\begin{aligned} \mathrm{U}(\xi) &= \chi^{1/4}\mu_{\mathrm{k}}(\mathrm{y}), \qquad \chi \equiv \xi'^2 = \frac{|\mathrm{g}(\mathrm{y})|}{\mathrm{f}^{(1)}(\xi)^2}, \\ \mathrm{f}(\xi) &= \int^{\mathrm{y}} \sqrt{|\mathrm{g}(\mathrm{y})|}\mathrm{dy}, \qquad \mathrm{f}^{(1)}(\xi) = \frac{\mathrm{df}(\xi)}{\mathrm{d}\xi}, \end{aligned} \tag{10}$$

- χ must be regular and not vanish in the intervals of interest
- $\mathrm{f}^{(1)}(\xi)$ must be chosen so that it has zeros and singularities of the same type as that of $\mathrm{g}(\mathrm{y})$

2.3 Uniform Asymptotic Approximation Method (Cont.)

- The equation of motion for the mode function reduces to,

$$\frac{d^2U(\xi)}{d\xi^2} = \left[\pm f^{(1)}(\xi)^2 + \psi(\xi)\right] U(\xi), \qquad (11)$$

with

$$\psi(\xi) = \frac{q(y)}{\chi} - \chi^{-3/4}\frac{d^2(\chi^{-1/4})}{dy^2}, \qquad (12)$$

in the above "$+$" for $g(y) > 0$, and "$-$" for $g(y) < 0$.

- Neglecting $\psi(\xi)$ we obtain solutions to the first-order approximation
- Choosing properly $f^{(1)}(\xi)$ in order for the equation to be:
(a) solved analytically, and
(b) minimizing the error bounds.

2.3 Uniform Asymptotic Approximation Method (Cont.)

- To illustrate our method, let us consider the case,

$$g(y) + q(y) \equiv -\frac{\Omega_k^2}{k^{-2}} = \frac{\nu^2 - 1/4}{y^2} - 1 + b_1\epsilon_*^2 y^2 - b_2\epsilon_*^4 y^4$$

- Approximate solution near turning points $g(y) = 0$:
  Assuming that y_0 is a single zero of $g(y) = 0$, we can choose

$$f^{(1)}(\xi)^2 = \pm\xi,$$

so that

$$\frac{d^2U(\xi)}{d\xi^2} = \Big(\xi + \psi(\xi)\Big)U(\xi)$$

2.3 Uniform Asymptotic Approximation Method (Cont.)

- Then, the first-order approximate solution,

$$\mathrm{U}(\xi) = \alpha_0\left(\mathsf{Ai}(\lambda^{2/3}\xi) + \epsilon_3^{(1)}\right) + \beta_0\left(\mathsf{Bi}(\lambda^{2/3}\xi) + \epsilon_4^{(1)}\right) \quad (13)$$

$\mathsf{Ai}(\xi)$, $\mathsf{Bi}(\xi)$: the Airy functions
$\epsilon_3^{(1)}$, $\epsilon_4^{(1)}$: the errors of the approximations

- The upper bounds of errors are

$$\frac{\left|\epsilon_3^{(1)}\right|}{\mathrm{M}(\xi)},\ \frac{|\partial\epsilon_3^{(1)}/\partial\xi|}{\mathrm{N}(\xi)} \leq \frac{\mathrm{E}^{-1}(\xi)}{\lambda}\left\{\exp\left(\lambda\mathscr{V}_{\xi,\mathrm{a}_3}(\mathrm{H})\right) - 1\right\},$$

$$\frac{\left|\epsilon_4^{(1)}\right|}{\mathrm{M}(\xi)},\ \frac{|\partial\epsilon_4^{(1)}/\partial\xi|}{\mathrm{N}(\xi)} \leq \frac{\mathrm{E}(\xi)}{\lambda}\left\{\exp\left(\lambda\mathscr{V}_{\mathrm{a}_4,\xi}(\mathrm{H})\right) - 1\right\}$$

2.3 Uniform Asymptotic Approximation Method (Cont.)

- where the error control function $H(y)$ defined as

$$H(\xi) = \int_{\xi}^{a_3} |v|^{-1/2}\psi(v)dv,$$

$$\psi(\xi) = \frac{q(y)}{\chi} - \chi^{-3/4}\frac{d^2(\chi^{-1/4})}{dy^2}$$

- Minimizing the error control function and thew upper bounds of errors requires the unique choice [17],

$$g(y) = \frac{\nu^2}{y^2} - 1 + b_1\epsilon_*^2 y^2 - b_2\epsilon_*^4 y^4,$$

$$q(y) = -\frac{1}{4y^2}.$$

[17] Zhu, AW, Cleaver, Kirsten, Sheng, PRD89 (2014) 043507; PRD93 (2016) 123525.

2.3 Uniform Asymptotic Approximation Method (Cont.)

- Then, the approximate solution of $U(\xi)$ up to the $(2n)$-th order of the approximation:

$$\begin{aligned}
U(\lambda,\xi) &= \alpha_0\left[\mathrm{Ai}(\lambda^{2/3}\xi)\sum_{s=0}^{n}\frac{A_s(\xi)}{\lambda^{2s}}\right. \\
&\qquad\left. + \frac{\mathrm{Ai}'(\lambda^{2/3}\xi)}{\lambda^{4/3}}\sum_{s=0}^{n-1}\frac{B_s(\xi)}{\lambda^{2s}} + \epsilon_3^{(2n+1)}\right] \\
&\quad +\beta_0\left[\mathrm{Bi}(\lambda^{2/3}\xi)\sum_{s=0}^{n}\frac{A_s(\xi)}{\lambda^{2s}}\right. \\
&\qquad\left. + \frac{\mathrm{Bi}'(\lambda^{2/3}\xi)}{\lambda^{4/3}}\sum_{s=0}^{n-1}\frac{B_s(\xi)}{\lambda^{2s}} + \epsilon_4^{(2n+1)}\right],
\end{aligned} \tag{14}$$

2.3 Uniform Asymptotic Approximation Method (Cont.)

- where

$$\mathrm{A}_0(\xi) = 1,$$

$$\mathrm{B}_\mathrm{s} = \frac{\pm 1}{2(\pm\xi)^{1/2}} \int_0^{\xi} \{\psi(\mathrm{v})\mathrm{A}_\mathrm{s}(\mathrm{v}) - \mathrm{A}_\mathrm{s}''(\mathrm{v})\} \frac{\mathrm{dv}}{(\pm \mathrm{v})^{1/2}},$$

$$\mathrm{A}_{\mathrm{s}+1}(\xi) = -\frac{1}{2}\mathrm{B}_\mathrm{s}'(\xi) + \frac{1}{2}\int \psi(\mathrm{v})\mathrm{B}_\mathrm{s}(\mathrm{v})\mathrm{dv},$$

where $\pm$ correspond to $\xi \geq 0$ and $\xi \leq 0$, respectively.

2.3 Uniform Asymptotic Approximation Method (Cont.)

- The upper bounds of the errors are:

$$
\begin{aligned}
&\frac{\epsilon_3^{(2n+1)}}{M(\lambda^{2/3}\xi)},\quad \frac{\partial\epsilon_3^{(2n+1)}/\partial\xi}{\lambda^{2/3}N(\lambda^{2/3}\xi)}\\
&\qquad \leq 2E^{-1}(\lambda^{2/3}\xi)\exp\left\{\frac{2\kappa_0\mathscr{V}_{\alpha,\xi}(|\xi^{1/2}|B_0)}{\lambda}\right\}\\
&\qquad\qquad \times \frac{\mathscr{V}_{\alpha,\xi}(|\xi^{1/2}|B_n)}{\lambda^{2n+1}},\\
&\frac{\epsilon_4^{(2n+1)}}{M(\lambda^{2/3}\xi)},\quad \frac{\partial\epsilon_4^{(2n+1)}/\partial\xi}{\lambda^{2/3}N(\lambda^{2/3}\xi)}\\
&\qquad \leq 2E(\lambda^{2/3}\xi)\exp\left\{\frac{2\kappa_0\mathscr{V}_{\xi,\beta}(|\xi^{1/2}|B_0)}{\lambda}\right\}\\
&\qquad\qquad \times \frac{\mathscr{V}_{\xi,\beta}(|\xi^{1/2}|B_n)}{\lambda^{2n+1}}.
\end{aligned}
\tag{15}
$$

2.3 Uniform Asymptotic Approximation Method (Cont.)

- We assume the universe was initially at the adiabatic (Bunch-Davies) vacuum,

$$\lim_{y\to+\infty}\mu_k(y) = \frac{1}{\sqrt{2\omega}}e^{-i\int\omega d\eta}$$

$$\simeq \sqrt{\frac{k}{2}}\frac{1}{(-g)^{1/4}}\exp\left(-i\int_{y_i}^{y}\sqrt{-g}dy\right).$$

- Since the equation of the mode function is second-order, we need one more condition to completely fix the free parameters in the solutions. We choose the second one as the Wronskian condition

$$\mu_k(y)\mu_k^*(y)' - \mu_k^*(y)\mu_k(y)' = i.$$

2.3 Uniform Asymptotic Approximation Method (Cont.)

- Comparison of numerical (exact) solution with our analytical approximate solution:

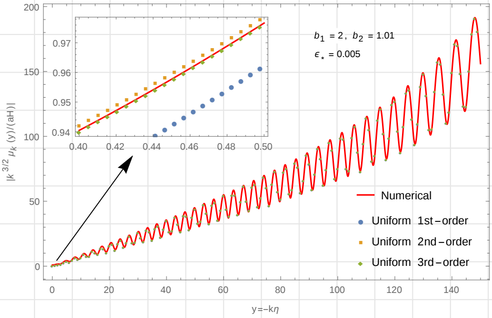

Figure: The numerical (exact) (red solid curves) and analytical (dotted curves) solutions with $b_1 = 3$, $b_2 = 2$, $\nu = 3/2$, and $\epsilon_* = H/M_* = 0.01$.

2.3 Uniform Asymptotic Approximation Method (Cont.)

- Up to the third-order approximations, the upper bounds of errors are

$$\leq 0.15\%$$

Table of Contents

3.1 Inflation With Quantum Gravitational Effects

- High-order power spectra and spectral indices have been calculated so far up to the second-order of the slow-roll parameters $\epsilon_{\rm n}$ in two cases:

  (a) GR ($\omega_{\rm k}^2 = {\rm k}^2$), first by using the Green function method and later confirmed by the improved WKB method[18],

  (b) k-inflation ($\omega_{\rm k}^2 = {\rm c}_{\rm s}^2(\eta){\rm k}^2$), by using the uniform approximation method but to the firs-order of $(1/\lambda)$[19].

[18] J.-O. Gong and E.D. Stewart, PLB510 (2001) 1; S.M. Leach, A, Liddle, J. Martin and D. Schwarz, PRD66 (2002) 023515; J.-O. Gong, CQG21 (2004) 5555; R. Casadio, et al, PRD71 (2005) 043517; PLB625 (2005) 1.

[19] J. Martin, C. Ringeval and V. Vennin, JCAP06 (2013) 021.

3.1 Inflation With Quantum Gravitational Effects (Cont.)

- Applying our method to GR up to the second-order in terms of the slow-roll parameters ϵ_n and the third-order in terms of λ, we find that the resulted power spectra and spectral indexes of both scalar and tensor perturbations are consistent with the ones obtained by the Green function and improved WKB methods[20], within the allowed errors [Zhu, AW, Cleaver, Kirsten, Sheng, PRD90 (2014) 063503].

[20] J.-O. Gong and E.D. Stewart, PLB510 (2001) 1; S.M. Leach, A. Liddle, J. Martin and D. Schwarz, PRD66 (2002) 023515; J.-O. Gong, CQG21 (2004) 5555; R. Casadio, et al, PRD71 (2005) 043517; PLB625 (2005) 1.

3.1 Inflation With Quantum Gravitational Effects (Cont.)

- Applying our method to k-inflation, we obtained the power spectra, spectral indexes and runnings of both scalar and tensor perturbations with the highest accuracy existing in the literature so far [21].
- Note that a large class of inflationary models belongs to k-inflation [22].

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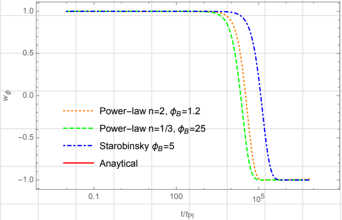
[21] Zhu, AW, Cleaver, Kirsten, Sheng, PRD90 (2014) 103517

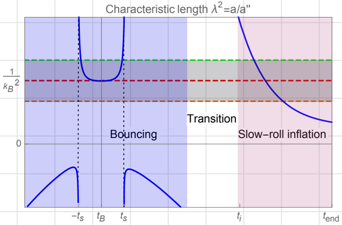
[22] J. Martin, C. Ringeval, and V. Vennin, Encyclopaedia Inflationaris, Phys. Dark Univ. 5 (2014) 75 [arXiv:1303.3787].

3.2 Quantum Gravitational Effects in LQC

- In LQC, the evolution of the background before reheating can be divided into three different phases:

Bouncing, transition, slow-roll inflation





Zhu, AW, Cleaver, Kirsten, Sheng, arXiv:16107.06329

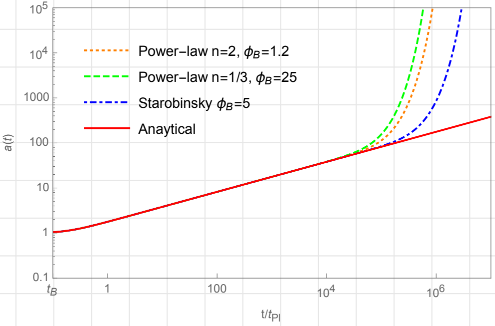
3.2 Quantum Gravitational Effects in LQC (Cont.)

- If the quantum bounce is dominated by kinetic energy of the inflaton, we found the evolution of the background is universal [23]

$$a(t) = a_B \left(1 + \gamma_B t^2/t_{Pl}^2\right)^{1/6}, \qquad (16)$$

$\gamma_B \equiv 24\pi\rho_c/m_{Pl}^4$

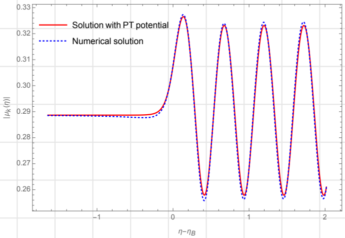
t_{Pl}: the Planck time.



[23] Zhu, AW, Cleaver, Kirsten, Sheng, arXiv:16107.06329

3.2 Quantum Gravitational Effects in LQC (Cont.)

- The scalar and tensor perturbations are all universal and independent of the slow-roll inflationary models in the pre-inflationary phase
- During the pre-inflationary phase, we find the mode functions μ_k's for the scalar and tensor perturbations analytically



3.2 Quantum Gravitational Effects in LQC (Cont.)

- Then, the Bogoliubov coefficients, α_k, β_k, at the onset of the slow-roll inflation can be read off explicitly and given by

$$\begin{aligned}
\alpha_k &= \Gamma(a_3)\Gamma(a_3 - a_1 - a_2)/[\Gamma(a_3 - a_1)\Gamma(a_3 - a_2)], \\
\beta_k &= e^{2ik\eta_B}\Gamma(a_3)\Gamma(a_1 + a_2 - a_3)/[\Gamma(a_1)\Gamma(a_2)], \\
a_1 &\equiv 1/2 + \sqrt{\alpha^2 - 4V_0}/(2\alpha) - ik/\alpha, \\
a_2 &\equiv 1/2 - \sqrt{\alpha^2 - 4V_0}/(2\alpha) - ik/\alpha, \\
a_3 &\equiv 1 - ik/\alpha
\end{aligned} \tag{17}$$

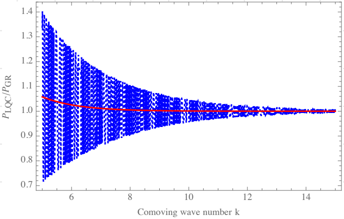
- Since $|\beta_k|^2 \neq 0$, particles are generically generated at the onset of inflation.

3.2 Quantum Gravitational Effects in LQC (Cont.)

- Note that in the standard inflationary scenario, the universe is in a vacuum at the onset of inflation,

$$\left|\alpha_k^{\mathrm{GR}}\right|^2 = 1, \quad \left|\alpha_k^{\mathrm{GR}}\right|^2 = 0$$

- Oscillations always happen in the power spectra, and their phases for both scalar and tensor perturbations are the same, in contrast to other theories of quantum gravity.



The main idea:

- In the Stage IV experiments, the accuracy of the measurements of the quantities $(n_s,\ r)$ are [24],

$$\sigma(n_s),\ \sigma(r) \simeq 10^{-3} \sim 10^{-4}$$

- In GR, we have

$$\Gamma\left(n_s, r\right) = 0. \tag{18}$$

For example, for the potential $V(\phi) = \lambda_n \phi^n$, we have

$$\Gamma_n(n_s, r) = (n_s - 1) + \frac{(2+n)r}{8n} + \frac{(3n^2 + 18n - 4)(n_s - 1)^2}{6(n+2)^2} = 0$$

---

[24] K. N. Abazajian et al., Astropart. Phys. 63, 55 (2015); arXiv: 1610.02743.

3.3 Detecting Quantum Gravitational Effects in the early Universe (Cont.)

- Then, the accuracy of the measurements of the quantity $\Gamma\left(n_s,\ r\right)$ is

$$\sigma\left[\Gamma\left(n_s, r\right)\right] \simeq 10^{-3} \sim 10^{-3}$$

- On the other hand, after quantum gravitational effects are taken into account, Eq.(18) is modified to the form

$$\Gamma_n(n_s, r) = \mathcal{F}_{QG}$$

- Clearly, if

$$\left|\mathcal{F}_{QG}\right| \gtrsim 10^{-4}$$

then, the Stage IV experiments is able to measure the quantum gravitational corrections $\mathcal{F}_{QG}$.

3.3 Detecting Quantum Gravitational Effects in the early Universe (Cont.)

- In the framework of deformed algebra approach, for $\sigma \leq 1$, we found that [25],

$$|\mathcal{F}_{\mathrm{QG}}| \gtrsim 10^{-3}$$

That is, it is within the range of detections in the current and forthcoming cosmological observations.

[25] Zhu, AW, Cleaver, Kirsten, Sheng, Wu, Astrophys. J. Lett. 807 (2015) L17; JCAP 10 (2015) 052; JCAP 03 (2016) 046.

3.3 Detecting Quantum Gravitational Effects in the early Universe (Cont.)

- It is remarkable to note that recently it was found that this approach is already inconsistent with current observations [26], as it produces too large

$$r_{\mathrm{D.A.}} > 0.2$$

- Current observations require [27],

$$r_{\mathrm{Obs.}} < 0.12 \quad (95\%\mathrm{C.L.})$$

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[26] B. Bolliet, A. Barrau, J. Grain, and S. Schander, Observational Exclusion of a Consistent Quantum Cosmology Scenario, PRD 93 (2016) 124011 (2016).

[27] Planck Collaboration, P. A. R. Ade et al., Planck 2015 results. XX. Constraints on inflation, arXiv:1502.02114.

3.3 Detecting Quantum Gravitational Effects in the early Universe (Cont.)

- Recently, we studied the problem in the framework of *dressed metric approach*, by using the Planck 2015 data [28] with the MCMC code developed by us some years ago [29].
- We vary the seven parameters,

$$\Omega_b h^2, \quad \Omega_c h^2, \quad \tau, \quad \Theta_s, \quad n_s, \quad A_s, \quad \frac{k_B}{a_0}$$

by using the high-l CMB temperature power spectrum (TT) and polarization data (TT, TE, EE) respectively with the low-l polarization data (lowP) from Planck2015.

[28] Planck Collaboration, P. A. R. Ade et al., Planck 2015 results. XX. Constraints on inflation, arXiv:1502.02114.

[29] Y.-G. Gong, Q. Wu, AW, Dark Energy and Cosmic Curvature: Monte Carlo Markov Chain Approach, Astrophys. J. 681, 27 (2008).

3.3 Detecting Quantum Gravitational Effects in the early Universe (Cont.)

- In the following Table, we list the best fit values of the six cosmological parameters and constraints on $\mathrm{k_B/a_0}$ and r at 95% C.L. ($\mathrm{k_B} \equiv \sqrt{8\pi\rho_B}\,\mathrm{a_B/m_{Pl}}$):

TABLE I. The Best fitting values of the six cosmological parameters and the constraints on k_B/a_0 and r at 95% C.L for different cosmological models from different data combinations.

Parameter	Planck TT+lowP	Planck TT,TE,EE+lowP	Planck TT+lowP+r	Planck TT,TE,EE+lowP+r
$\Omega_b h^2$	0.022355	0.022193	0.022322	0.022064
$\Omega_c h^2$	0.11893	0.12000	0.11908	0.12071
$100\theta_{MC}$	1.04115	1.04065	1.04080	1.04057
τ	0.077835	0.089272	0.081955	0.085259
$\ln(10^{10}A_s)$	3.088	3.112	3.101	3.104
n_s	0.9662	0.9647	0.9658	0.9607
k_B/a_0	$< 3.12\times10^{-4}$	$< 3.05\times10^{-4}$	$< 3.14\times10^{-4}$	$< 3.14\times10^{-4}$
r	$----$	$----$	< 0.113	< 0.107

Zhu, AW, Cleaver, Kirsten, Sheng, arXiv:16107.06329

3.3 Detecting Quantum Gravitational Effects in the early Universe (Cont.)

- It is clear that the theory is consistent with observations, provided that

$$\frac{k_B}{a_0} < 3.14 \times 10^{-4}\,\mathrm{Mpc}^{-1} (95\%\mathrm{C.L.})$$

- Taking $\rho_B = 0.41 m_{Pl}^4$ [30], we find that the above constraint implies that

$$N_{tot} \equiv \ln \frac{a_0}{a_B} > 132 \;\; (95\%\mathrm{C.L.})$$

[30] I. Agullo, A. Ashtekar, and W. Nelson, Quantum Gravity Extension of the Inflationary Scenario, PRL109 (2012) 251301.

Table of Contents

4. Concluding Remarks

- To establish a proper theory of Quantum Gravity, experimental evidences of their effects are highly demanded.
- Quantum gravitational effects in the early universe become important and need to be taken into account
- With the arrival of the era of precision cosmology, it becomes possible to test different theories of quantum gravity by cosmological observations
- The uniform asymptotic approximation method is specially designed for this purpose
- To its third-order approximations, the upper bounds of errors are

$$\lesssim 0.15\%$$

which is sufficient for current and forthcoming observations

5. Conclusions (Cont.)

- In LQC, *the dressed metric approach* is consistent with current observations, provided that

$$\frac{k_B}{a_0} < 3.14 \times 10^{-4} \text{Mpc}^{-1} (95\% \text{C.L.})$$

- Right now, we are studying quantum gravitational effects from other theories of gravity, including string/M-Theory [31] and Horava-Lifshitz gravity [32].

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[31] D. Baumann and L. McAllister, Inflation and String Theory (Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2015).

[32] AW, Horava Gravity at a Lifshitz Point: A Progress Report, Inter. J. Mod. Phys. D26 (2017) 1730014 [arXiv:1701.06087].

Thank You!