Null Hypersurfaces in Space-Times of Constant Curvature

Peter A. Hogan

School of Physics
University College Dublin
Ireland

May 7, 2017

Space-Times of Constant Curvature

$$R_{abcd} = K\,(g_{ac}\,g_{bd} - g_{ad}\,g_{bc})\ ,\quad K = \mathrm{constant} \neq 0$$

$$\Rightarrow\ R_{ab} = -\Lambda\,g_{ab}\quad ,\quad \Lambda = 3\,K$$

$$\Rightarrow\ C_{abcd} = 0$$

$\Rightarrow$ coordinates $x^a = (x, y, z, t)$ exist such that $g_{ab} = \lambda^2\eta_{ab}$ with $\eta_{ab} = \mathrm{diag}(1, 1, 1, -1)$ and

$$\lambda = \left(1 + \frac{\Lambda}{12}\eta_{ab}\,x^a\,x^b\right)^{-1}$$

Note: cannot have a k_a for which $k_{a;b} = 0$ if $K \neq 0$ since

$$0 = k_{a;bc} - k_{a;cb} = k_d\,R^d{}_{abc} = K\,(k_b\,g_{ac} - k_c\,g_{ab})$$

Null Hyperplanes

- A null hyperplane is a null hypersurface generated by null geodesics which are shear-free and expansion-free
- *Null*, *Geodesic*, *Shear-Free* are conformally invariant properties
- All shear–free null hypersurfaces in Minkowskian space-time are known [they are either null hyperplanes (Case 1) or null cones (Case 2) or portions thereof]

Given g_{ab} and k^a such that $g_{ab}\,k^a\,k^b = 0$; k^a is geodesic and shear-free if and only if there exists ξ_a and $\varphi(x^a)$ such that

$$k_{a;b} + k_{b;a} = \xi_a\,k_b + \xi_b\,k_a + \varphi\,g_{ab}$$

[I. Robinson and A. Trautman, J. Math. Phys. **24**, 1425 (1983)]

$$k^a \quad \text{expansion} - \text{free} \quad \Leftrightarrow \quad k^a{}_{;a} = 0$$

Null Hyperplanes

- A null hyperplane is a null hypersurface generated by null geodesics which are shear-free and expansion-free
- *Null*, *Geodesic*, *Shear-Free* are conformally invariant properties
- All shear–free null hypersurfaces in Minkowskian space-time are known [they are either null hyperplanes (Case 1) or null cones (Case 2) or portions thereof]

Given g_{ab} and k^a such that $g_{ab}\,k^a\,k^b = 0$; k^a is geodesic and shear-free if and only if there exists ξ_a and $\varphi(x^a)$ such that

$$k_{a;b} + k_{b;a} = \xi_a\,k_b + \xi_b\,k_a + \varphi\,g_{ab}$$

[I. Robinson and A. Trautman, J. Math. Phys. **24**, 1425 (1983)]

$$k^a \quad \text{expansion} - \text{free} \quad \Leftrightarrow \quad k^a{}_{;a} = 0$$

Null Hyperplanes

- A null hyperplane is a null hypersurface generated by null geodesics which are shear-free and expansion-free
- *Null, Geodesic, Shear-Free* are conformally invariant properties
- All shear–free null hypersurfaces in Minkowskian space-time are known [they are either null hyperplanes (Case 1) or null cones (Case 2) or portions thereof]

Given g_{ab} and k^a such that $g_{ab}\,k^a\,k^b = 0$; k^a is geodesic and shear-free if and only if there exists ξ_a and $\varphi(x^a)$ such that

$$k_{a;b} + k_{b;a} = \xi_a\,k_b + \xi_b\,k_a + \varphi\,g_{ab}$$

[I. Robinson and A. Trautman, J. Math. Phys. **24**, 1425 (1983)]

$$k^a \quad \text{expansion} - \text{free} \quad \Leftrightarrow \quad k^a{}_{;a} = 0$$

Minkowskian Space-Time

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 = \eta_{ab}\, dx^a\, dx^b$$

Shear-free null hypersurfaces are given by $u(x, y, z, t) = \text{constant}$ with $u(x, y, z, t)$ defined implicitly by

Case 1 (Null Hyperplane):

$$\eta_{ab}\, a^a(u)\, x^b + b(u) = 0 \quad \text{with} \quad \eta_{ab}\, a^a\, a^b = 0$$

Case 2 (Null Cone):

$$\eta_{ab}\, (x^a - w^a(u))\, (x^b - w^b(u)) = 0$$

Case 1 (η_{ab})

$$\text{Case 1} \Rightarrow \quad u_{,b} = -f^{-1}a_b \ \text{ with } \ f = \dot{b} + \dot{a}_b\, x^b$$

$$\Rightarrow \quad u = \text{constant are null}$$

$$u_{,b,c} = -f^{-1}(\dot{a}_b\, u_{,c} + \dot{a}_c\, u_{,b}) - f^{-1}\dot{f}\, u_{,b}\, u_{,c}$$

$$\Rightarrow \quad u_{,a} \text{ is geodesic and shear} - \text{free}$$

$$\eta^{bc}\, u_{,b,c} = 2\, f^{-2}\eta^{bc}\, \dot{a}_b\, a_c = 0$$

$$\Rightarrow \quad u_{,a} \text{ is expansion} - \text{free}$$

* * *

$u = \text{constant}$ are confirmed as null hyperplanes in Minkowskian space-time. What is the condition that they are null hyperplanes in the space-time with metric $g_{ab} = \lambda^2\eta_{ab}$?

Case $1 (g_{ab} = \lambda^2 \eta_{ab})$

The components of the Riemannian connection associated with $g_{ab} = \lambda^2 \eta_{ab}$ are

$$\Gamma^a_{bc} = \lambda^{-1}(\lambda_{,b}\, \delta^a_c + \lambda_{,c}\, \delta^a_b - \eta^{ad}\, \lambda_{,d}\, \eta_{bc})$$

$$\lambda^{-1} = 1 + \frac{\Lambda}{12}\eta_{ab}\, x^a\, x^b \quad \text{and} \quad \lambda_{,a} = -\lambda^2 \Lambda\, \eta_{ab}\, x^b/6$$

$$\Rightarrow \quad u_{,b;c} = \xi_b\, u_{,c} + \xi_c\, u_{,b} + \lambda^{-1}\eta^{ad}\, \lambda_{,a}\, u_{,d}\, \eta_{bc}$$

with

$$\xi_b = -f^{-1}\dot{a}_b - \lambda^{-1}\lambda_{,b} - \frac{1}{2}f^{-1}\dot{f}\, u_{,b}$$

$$\Rightarrow \;\; \eta^{bc}\, u_{,b;c} = -2\, \lambda^{-1} f^{-1}\eta^{bc}\, \lambda_{,b}\, a_c = \frac{\Lambda}{3}\lambda\, f^{-1} a_b\, x^b = -\frac{\Lambda}{3}\lambda\, f^{-1} b(u)$$

(remember $a_c(u)\, x^c + b(u) = 0$)

Conclusion: The null hyperplanes $u = \text{constant}$ in Minkowskian space-time given by $a_c(u)\, x^c + b(u) = 0$ correspond to null hyperplanes in the space-times of constant curvature if $b(u) = 0$.

$a_c(u)\,x^c = 0$

The null hyperplanes $u = \text{constant} \;\Leftrightarrow\; a_c(u)\,x^c = 0$ intersect each other. Only the *direction* of a^c is significant and is determined by an arbitrary complex-valued function $l(u)$ with complex conjugate $\bar{l}(u)$ via

$$a^1 + ia^2 = 2\sqrt{2}\,l(u)\;,\;\; a^3 + a^4 = 4\,l(u)\,\bar{l}(u)\;,\;\; a^3 - a^4 = -2$$

Now

$$a_c(u)\,x^c = 0 \quad\Leftrightarrow\quad z + t = \sqrt{2}\,\bar{l}(u)\,(x+iy) + \sqrt{2}\,l(u)\,(x-iy) + 2\,l(u)\,\bar{l}(u)\,(z-t)$$

$$\Rightarrow\quad \eta_{ab}\,x^a\,x^b = \left|x + iy + \sqrt{2}\;l(u)\,(z-t)\right|^2$$

Temptation! Define

$$\zeta = \frac{1}{\sqrt{2}}(x+iy) + l(u)\,(z-t)$$

and use $\zeta, \bar{\zeta}, u, z-t$ as coordinates satisfying

$$x + iy = \sqrt{2}\,\zeta - \sqrt{2}\,l(u)\,(z-t)\ ,$$

$$z + t = 2\{\bar{l}(u)\,\zeta + l(u)\,\bar{\zeta}\} - 2\,l(u)\,\bar{l}(u)\,(z-t)\ ,$$

$$\lambda^{-1} = 1 + \frac{\Lambda}{12}\eta_{ab}\,x^a\,x^b = 1 + \frac{\Lambda}{6}\zeta\,\bar{\zeta} = p\ (\text{say})$$

$\Rightarrow$

$$ds^2 = \lambda^2\eta_{ab}\,dx^a\,dx^b = 2\,p^{-2}d\zeta\,d\bar{\zeta} + 2\,p^{-2}du \times \Sigma$$

with the 1-form

$$\Sigma = -(z-t)(\beta\,d\bar{\zeta} + \bar{\beta}\,d\zeta) + (\beta\,\bar{\zeta} + \bar{\beta}\,\zeta)(dz - dt) + \beta\,\bar{\beta}(z-t)^2 du$$

where $\beta(u) = dl(u)/du$. Define

$$q = \beta\,\bar{\zeta} + \bar{\beta}\,\zeta$$

and in place of $z - t$ a new coordinate r by

$$z - t = q\,r$$

The line element takes the Ozsváth-Robinson-Rózga [J. Math. Phys. **26**, 1755 (1985)] form

$$ds^2 = 2\, p^{-2} d\zeta\, d\bar{\zeta} + 2\, p^{-2} q^2 du\{dr + (q^{-1}\dot{q}\, r + \beta\,\bar{\beta}\, r^2)du\}$$

with

$$p = 1 + \frac{\Lambda}{6}\zeta\,\bar{\zeta}\ ,\ \ q = \beta\,\bar{\zeta} + \bar{\beta}\,\zeta\ ,\ \ \dot{q} = \frac{\partial q}{\partial u}$$

where $\beta(u) = dl(u)/du$.

Observation: This de Sitter (if $\Lambda > 0$) or anti de Sitter (if $\Lambda < 0$) line element is expressed in a coordinate system based on a family $u = \mathrm{constant}$ of null hyperplanes. It contains two arbitrary real-valued functions of u (the real and imaginary parts of $\beta(u)$) and we know the geometrical origin of them!

Case 2 (η_{ab})

$$\eta_{ab}\,(x^a - w^a(u))(x^b - w^b(u)) = 0 \quad \Rightarrow \quad u_{,a} = \frac{\mu_a}{R}$$

$$\mu^a = x^a - w^a(u)\ , \quad R = \eta_{ab}\,\dot{w}^a(u)\,\mu^b \quad (\Rightarrow \quad \dot{w}^a\,u_{,a} = +1)$$

confirming that $u = \text{constant}$ are null hypersurfaces

$$\mu^a{}_{,b} = \delta^a_b - \dot{w}^a\,u_{,b} \ \ \text{and} \ \ R_{,b} = \dot{w}_b + A\,u_{,b} \ \ (A = -\dot{w}_a\,\dot{w}^a + R\,\ddot{w}^a\,u_{,a})$$

$\Rightarrow$

$$u_{,ab} = \frac{1}{R}(\eta_{ab} - \dot{w}_a\,u_{,b} - \dot{w}_b\,u_{,a} - A\,u_{,a}\,u_{,b})$$

confirming that $u_{,a}$ is geodesic and shear-free with expansion $\eta^{ab}\,u_{,ab}/2 = 1/R \neq 0$

Case 2 ($g_{ab} = \lambda^2 \eta_{ab}$)

In the space-time of constant curvature

$$u_{,a;b} = \xi_a\, u_{,b} + \xi_b\, u_{,a} + (R^{-1} + \lambda^{-1}\eta_{cd}\,\lambda_{,c}\,u_{,d})\,\eta_{ab}$$

with

$$\xi_a = -R^{-1}\dot{w}_a + \lambda^{-1}\lambda_{,a} - \frac{1}{2}A\,R^{-1}u_{,a}$$

confirming that $u_{,a}$ is geodesic and shear-free. $u_{,a}$ is expansion–free if

$$\eta^{ab}u_{,a;b} = 0 \;\Leftrightarrow\; R^{-1} + \lambda^{-1}\eta^{ab}\,\lambda_{,a}\,u_{,b} = 0 \;\Leftrightarrow\; \eta_{ab}\,w^a(u)\,w^b(u) = -\frac{12}{\Lambda}$$

Given Λ we see that $w^a(u)$ has three independent components which can be parametrized with a real-valued function $m(u)$ and a complex-valued function $l(u)$ and its complex conjugate $\bar{l}(u)$ as:

$\eta_{ab}\, x^a\, x^b - 2\,\eta_{ab}\, x^a\, w^b(u) = 12/\Lambda$

$$w^1+iw^2 = \frac{6\sqrt{2}\,l(u)}{\Lambda\, m(u)}\ ,\ w^3+w^4 = \frac{6}{\Lambda\, m(u)}\left(\frac{1}{3}\Lambda\, m^2(u) + 2\,l(u)\,\bar{l}(u)\right)\ ,$$

$$w^3 - w^4 = -\frac{6}{\Lambda\, m(u)}$$

We assume $m(u) \neq 0$ but for small $m(u)$ the equation above for $u = \text{constant}$ approximates $a_b(u)\, x^b = 0$ with

$$a^1 + ia^2 = 2\sqrt{2}\, l(u)\ ,\ a^3 + a^4 = 4\, l(u)\,\bar{l}(u)\ ,\ a^3 - a^4 = -2$$

The equation above for $u = \text{constant}$ reads:

$$z + t = \sqrt{2}\,\bar{l}(u)(x+iy) + \sqrt{2}\, l(u)(x-iy) + 2\, l(u)\,\bar{l}(u)(z-t)$$
$$+2\, m(u)\left(1 + \frac{\Lambda}{6} m(u)\,(z-t)\right) - \frac{\Lambda}{6}\, m(u)\,\eta_{ab}\, x^a\, x^b$$

$\Rightarrow$

$$\eta_{ab}\, x^a\, x^b = \left(1 + \frac{\Lambda}{6}\, m(u)\,(z-t)\right)^{-1} \left|x + iy + \sqrt{2}\, l(u)\,(z-t)\right|^2$$

$$+2\, m(u)\,(z-t)$$

With $\lambda^{-1} = 1 + \frac{\Lambda}{12}\eta_{ab}\, x^a\, x^b$ we now calculate

$$\left(1 + \frac{\Lambda}{6}\, m(u)\,(z-t)\right)\, \lambda^{-1} = \left(1 + \frac{\Lambda}{6}\, m(u)\,(z-t)\right)^2$$

$$+\frac{\Lambda}{12}\, \left|x + iy + \sqrt{2}\, l(u)\,(z-t)\right|^2$$

Temptation! Define

$$\zeta = \left(1 + \frac{\Lambda}{6}\, m(u)\,(z-t)\right)^{-1} \left\{\frac{1}{\sqrt{2}}(x+iy) + l(u)\,(z-t)\right\}$$

$\eta_{ab}\, x^a\, x^b - 2\, \eta_{ab}\, x^a\, w^b(u) = 12/\Lambda$

$\Rightarrow$

$$\lambda^{-1} = \left(1 + \frac{\Lambda}{6}\, m(u)\,(z-t)\right)\, p \quad \text{with} \quad p = 1 + \frac{\Lambda}{6}\zeta\,\bar{\zeta}$$

Use $\zeta, \bar{\zeta}, u, z-t$ as coordinates with

$$x + iy = \sqrt{2}\,\zeta\, \left(1 + \frac{\Lambda}{6} m(u)\,(z-t)\right) - \sqrt{2}\, l(u)\,(z-t)$$

and

$$z+t = 2\,\left\{\bar{l}(u)\,\zeta + l(u)\,\bar{\zeta} + m(u)\,\left(1 - \frac{\Lambda}{6}\zeta\,\bar{\zeta}\right)\right\}\left(1 + \frac{\Lambda}{6} m(u)\,(z-t)\right)$$

$$-\left(\frac{\Lambda}{3}\, m^2(u) + 2\, l(u)\,\bar{l}(u)\right)(z-t)$$

Introducing a coordinate r via

$$z - t = \left(1 + \frac{\Lambda}{6} m(u)\,(z - t)\right) q\, r$$

with

$$q(\zeta, \bar{\zeta}, u) = \beta(u)\, \bar{\zeta} + \bar{\beta}(u)\, \zeta + \alpha(u)\, \left(1 - \frac{\Lambda}{6} \zeta \bar{\zeta}\right)$$

$[\beta(u) = dl(u)/du, \alpha(u) = dm(u)/du]$ The line element in coordinates $\zeta, \bar{\zeta}, r, u$ reads

$$ds^2 = \lambda^2 \eta_{ab}\, dx^a\, dx^b = 2\, p^{-2} d\zeta\, d\bar{\zeta} + 2\, p^{-2} q^2 du\{dr + (q^{-1} \dot{q}\, r + \frac{1}{2}\, \kappa\, r^2) du\}$$

with

$$\kappa = \frac{\Lambda}{3}\, \alpha^2(u) + 2\, \beta(u)\, \bar{\beta}(u)$$

Observation:

We have arrived at the general Ozsváth-Robinson-Rózga form of the de Sitter (if $\Lambda > 0$) or anti de Sitter (if $\Lambda < 0$) line element expressed in a coordinate system based on a family $u = \mathrm{constant}$ of null hyperplanes. It contains three arbitrary real-valued functions of u ($\alpha(u)$ and the real and imaginary parts of $\beta(u)$) and we know the geometrical origin of them!

Question:

What is the geometrical role of the functions $\alpha(u)$ and $\beta(u)$ in the space-times of non-zero constant curvature?

Intersecting Null Hyperplanes

When viewed in Minkowskian space-time the null hyperplanes

$$a_b(u)\,x^b = 0$$

intersect. When viewed in Minkowskian space-time the null hyperplanes

$$\eta_{ab}\,x^a\,x^b - 2\,\eta_{ab}\,x^a\,w^b(u) = 12/\Lambda$$

are null cones with vertices on the world line $x^a = w^a(u)$ with

$$\eta_{ab}\,w^a(u)\,w^b(u) = -\frac{12}{\Lambda}$$

The null cones intersect if the word line is space-like or time-like. They also intersect if the world line is null except when it is a common generator of the null cones.

Intersecting Null Hyperplanes

With $w^a(u)$ given in terms of $l(u), m(u)$ and Λ we find that

$$\eta_{ab}\,\dot{w}^a\,\dot{w}^b = \left(\frac{6}{\Lambda\, m}\right)^2 \kappa \quad \text{with} \quad \kappa = \frac{\Lambda}{3}\,\alpha^2 + 2\,\beta\,\bar{\beta}$$

For $x^a = w^a(u)$ a null geodesic we must have

$$\kappa = 0 \quad \text{and} \quad \ddot{w}^a = C(u)\,\dot{w}^a$$

for some $C(u)$. The information we can extract from these equations is encapsulated in:

$$\kappa = 0$$

and

$$\frac{1}{\beta}\,\frac{d\beta}{du} = \frac{1}{\bar{\beta}}\,\frac{d\bar{\beta}}{du} \;\Rightarrow\; \mathrm{Re}\beta = 0 \text{ or } \mathrm{Im}\beta = 0 \text{ or } \mathrm{Re}\beta = c_0\,\mathrm{Im}\beta$$

for some real number c_0.

Robinson-Tran Theorem

(1) $\Lambda > 0 \;\Rightarrow\; \kappa > 0 \;\Rightarrow$ intersecting null hyperplanes

(2) $\Lambda < 0 \;\Rightarrow\; \kappa > 0$ or $\kappa < 0$ or $\kappa = 0$ with

(i) $\kappa > 0 \;\Rightarrow$ intersecting null hyperplanes

(ii) $\kappa < 0 \;\Rightarrow$ intersecting null hyperplanes

(iii) $\kappa = 0 \;\Rightarrow$ intersecting null hyperplanes *except* when
$\mathrm{Re}\beta = 0$ or $\mathrm{Im}\beta = 0$ or $\mathrm{Re}\beta = c_0\,\mathrm{Im}\beta$

I. Robinson, University of Texas at Dallas Internal Report (1983);
H. V. Tran, Ph.D. thesis, University of Texas at Dallas (1988).

Andrzej Trautman:

"Gravitational waves are usually defined by their geometric properties. There is another important property of such waves, both linear and gravitational: *waves can propagate information*. This means that wave-like solutions depend on arbitrary functions, the shape of which contains the information carried by the wave"

Gravitational Waves with $\Lambda \neq 0$

Following Trautman's example we modify the de Sitter or anti-de Sitter line elements above to read:

$$ds^2 = 2\,p^{-2}d\zeta\,d\bar{\zeta} + 2\,p^{-2}q^2du\{dr + (q^{-1}\dot{q}\,r + \frac{1}{2}\,\kappa\,r^2 + F(\zeta,\bar{\zeta},u))du\}$$

with F arbitrary in u. Putting $F = p\,q^{-1}H(\zeta,\bar{\zeta},u)$ we have

$$R_{ab} = -\Lambda\,g_{ab} \quad \Rightarrow \quad H_{\zeta\bar{\zeta}} + \frac{\Lambda}{3}p^{-2}H = 0$$

The only non-vanishing null tetrad component of the Weyl tensor is

$$C_{abcd}\,l^a\,m^b\,l^c\,m^d = p^2q^2\frac{\partial}{\partial\zeta}\left(q^2\frac{\partial}{\partial\zeta}(p\,q^{-1}H)\right)\ ,\ \ k_a\,dx^a = p^{-1}q\,du$$

We have here the Ozsváth-Robinson-Rózga plane-fronted gravitational waves with $\Lambda \neq 0$