Problems and techniques in low-dimensional topology; or, how I became a knot theorist

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Freedman imported the theory of higher-dimensional manifolds to the study of 4-manifolds up to homeomorphism, combining aspects of Bing's point set perspective with the algebraic topology, especially "surgery theory" pioneered by Bill Browder.

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Idea: study the solution space to the **anti-self-dual Yang-Mills equations**. Topological properties of this solution space to a non-linear elliptic partial differential equation (depending on a choice of Riemannian metric!) can be used to explore differential topology of the underlying space.

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The existence follows from a combination of Freedman's and Donaldson's theories. In fact, Clifford Taubes showed that there are uncountably many such manifolds. I will sketch the construction of one. This description will hinge on *knot theory.*

Studying knots

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CONJECTURE $u(K_1 \# K_2) = u(K_1) + u(K_2)$

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There were many reproofs of this since; I will sketch one here a little later. But first, let's discuss some other knot invariants.

The Seifert genus

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There are non-trivial knots with $g_s = 0$; e.g.

$$K' = K \# \mathrm{mirr}(-K).$$

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The above characterization is due to John Conway

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- It is but a polynomial: it has limited algebraic structure.
- There are many knots it cannot distinguish: in fact, many knots have $\Delta_K(t) = 1$.

Knot Floer homology

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In fact, there is an even better version

$$\mathrm{HFK}^-(K) = \bigoplus_{d,s\in\mathbb{Z}} \widehat{\mathrm{HFK}}_m(K,s),$$

structure of a module over the polynomial algebra $\mathbb{F}[U]$ where

$$U\colon \mathrm{HFK}^-_d(K,s) \to \mathrm{HFK}^-_{d-2}(K,s-1)$$

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Knot Floer homology refines the Alexander polynomial

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Knot Floer homology detects the Seifert genus

THEOREM (Ozsváth-Szabó, 2003)

$$g(K) = \max\{s \big| \exists d, \widehat{\mathrm{HFK}}_d(K,s) \neq 0\}.$$

In particular, $\dim \widehat{\mathrm{HFK}}(K) = 1$ iff $K =$ the unknot.

A numerical invariant from Knot Floer homology

Let

$$\tau(K) = -\max\{s \big| \exists \xi \in \mathrm{HFK}^-(K,s) \text{ so that } U^m \cdot \xi \neq 0 \ \forall m \geq 0\}$$

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- $\tau(O) = 0$.
- $|\tau(K_+) - \tau(K_-)| \leq 1$; and therefore $|\tau(K)| \leq u(K)$.
- $\tau(T_{p,q}) = \frac{(p-1)(q-1)}{2}$; so the Milnor conjecture follows immediately.
- $|\tau(K)| \leq g_s(K)$.

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- (Freedman, 1980) Any knot $K \subset S^3$ with $\Delta_K(t) \equiv 1$ is topologically slice.
- (Freedman amd Quinn, 1990) Any connected, non-compact manifold admits a smooth structure.

Construction of an "exotic $\mathbb{R}^4$"

- If $K \subset S^3$ is topologically slice, then there is a smooth four-manifold $R_K \simeq \mathbb{R}^4$ so that X_K embeds smoothly in R_K.

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- If X_K embeds smoothly into $\mathbb{R}^4$, then K is smoothly slice. Take a singular surface that embeds, and consider a complement of the singularity.

Construction of the exotic $\mathbb{R}^4$

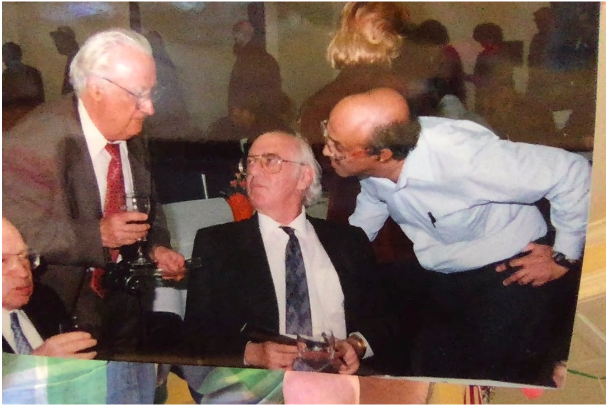
Thus, we must exhibit a knot that is topologically slice but not smoothly slice. Consider the Whitehead double of $T_{2,3}$. It is topologically slice by Freedman's theorem (since $\Delta_K(t) = 1$). It has $\tau(K) = 1$, by direct computation.

Three constructions of knot Floer homology

- The original construction uses the theory of pseudo-holomorphic curves.
- In 2006, a combinatorial formulation was discovered by Ciprian Manolescu, Sucharit Sarkar, and me.
- Right now, Szabó and I are developing an algebraic formulation, which is much more computable.

I do own a tie (or I did in 1998)

Ivor, my father, and Ali Hooshyar (with Yuval Ne'eman)



Engelbert Schucking, Ne'eman, Roger Penrose

Thank you