Time after Time: An Historical Perspective on the Problem of Time in Quantum Gravity

Donald Salisbury

Austin College

May 1, 2011

Overview

1. My constrained perspective
2. Léon Rosenfeld's constrained Hamiltonian formalism of 1930
3. Rosenfeld - Dirac puzzles
4. Peter Bergmann and collaborators
5. Significance of diffeomorphism invariance in alternative approaches: covariant quantization and path integrals
6. Paul Dirac
7. Arnowitt, Deser, and Misner
8. Frozen time
9. Pons, Salisbury, Shepley and Sundermeyer
   - The diffeomorphism-induced transformation group
   - Construction of diffeomorphism invariants

Growing up with Peter Bergmann

Figure: Max Bergmann with son Peter

Figure: Peter Bergmann's aunt, Clara Grunwald. Founder of the German Montessori movement

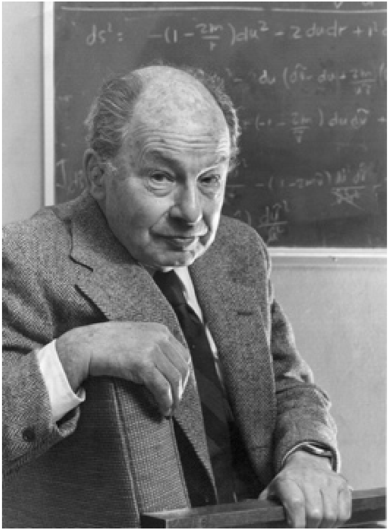

Figure: Peter Bergmann in 1982

Rosenfeld and Pauli

"I came to Zürich before the summer semester I came from Göttingen where I was still at the time. I had already corresponded with Bohr, asking him whether I could come to Copenhagen and so I wrote to Pauli then to ask him if he would take me up. He was very friendly and he said: With pleasure, because we have just completed a scheme of quantum electrodynamics with Heisenberg; dass ist ein Gebiet, dass noch nicht abgebrochen ist. So he was eager to have people brush up the details and explore the consequences and that is what I did at Zürich actually" (AHQM 7/19/63, p 5)

" I got provoked by Pauli to tackle this problem of the quantization of gravitation and the gravitation effects of light quanta, which at that time were more interesting. When I explained to Pauli what I wanted to work out, I think it was the Kerr effect or some optical effect, he said Well, you may do that, and I am glad beforehand for any result you may find. That was a way of saying that this was a problem that was not instructive, that any result might come out, whereas at that time, the calculation of the self energy of the light quantum arising from its gravitational field was done with a very definite purpose." (AHQM 7/19/63, p 8)

"Then Pauli told me that he was not at all pleased with longitudinal waves, so he wanted to have them treated another way, which I did, but that was not more enlightening, far from it." (AHQM 7/19/63, p 9)

" There was this point in their proof in which the invariants of the Hamiltonian seemed to depend on a special structure of the Hamiltonian, and that looked suspicious Yes, I understand that [said Pauli], but we have not been able to find a mistake in our calculation and we do not understand what this means; we suspect that it must be wrong, but we dont know. Then the thing came to a crisis through the fact that I tried to make a more general formulation of field quantization It was a purely abstract scheme which worked in a completely general way with only this complication of accessary conditions, but at any rate, not due to any special structure but only to the existence of invariance with respect to a group. So at that stage I was convinced that there must be a mistake in the original paper (AHQM 7/19/63, p 5)

“As I was investigating these relations in the especially instructive example of gravitation theory, Professor Pauli helpfully indicated to me the principles of a simpler and more natural manner of applying the Hamiltonian procedure in the presence of identities. This procedure is not subject to the disadvantages of the earlier methods.

Translation and commentaries

- Translation and Commentary of Léon Rosenfelds Zur Quantelung der Wellenfelder, Max Planck Institute for the History of Science Preprint 381
- Léon Rosenfeld and the challenge of the vanishing momentum in quantum electrodynamics, *Studies in History and Philosophy of Modern Physics* 40 (2009) 363
- Léon Rosenfeld's pioneering steps toward a quantum theory of gravity, J. Phys.: Conf. Ser. 222 (2010) 012052

Rosenfeld's coordinate transformations

Considers infinitesimal coordinate transformations $x'^{\mu} = x^{\mu} + \delta x^{\nu}$, where

$$\delta x^{\nu} = a_r^{\nu,0}(x)\epsilon^r(x) + a_r^{\nu,\sigma}(x)\frac{\partial \epsilon^r}{\partial x^{\sigma}} + \cdots ,$$

Assumes variables Q_{α} transform as

$$\delta Q_{\alpha} = c_{\alpha r}(x, Q)\epsilon^r(x) + c_{\alpha r}^{\sigma}(x, Q)\frac{\partial \epsilon^r}{\partial x^{\sigma}} + c_{\alpha r}^{\sigma\cdots\tau}(x, Q)\frac{\partial^j \epsilon^r}{\partial x^{\sigma}\cdots\partial x^{\tau}}.$$

Free relativistic particle example: $q^{\mu}(t)$ is parameterized spacetime position. Under the infinitesimal reparameterization

$$t' = t - \epsilon(t)$$

$$\delta q^{\mu} := q'^{\mu}(t') - q^{\mu}(t) = 0,$$

and

$$\delta N = \dot{\epsilon} N,$$

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Rosenfeld Lagrangian

Considers Lagrangians that are quadratic in field derivatives:

$$\mathcal{L} = \frac{1}{2}\Big(Q_{\alpha,\nu}\mathcal{A}^{\alpha\nu,\beta\mu}(Q)Q_{\beta,\mu}$$
$$+Q_{\alpha,\nu}\mathcal{B}^{\alpha\nu}(Q) + \mathcal{B}^{\alpha\nu}(Q)Q_{\alpha,\nu}$$
$$+\mathcal{C}(Q)\Big).$$

Assumes that the δQ_α are symmetry transformations so that the Lagrangian transforms as a scalar density of weight one:

$$\delta\mathcal{L} + \mathcal{L}\frac{\partial \delta x^\mu}{\partial x^\mu} \equiv 0. \qquad (1)$$

Free relativistic particle example:

$$L = \frac{1}{2N}\eta_{\mu\nu}\dot{q}^\mu\dot{q}^\nu - \frac{1}{2}m^2c^2N$$

Under $\delta t = \epsilon(t)$,

$$\delta L + L\frac{\partial \epsilon}{\partial t} \equiv 0.$$

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The origin of constraints

Canonical momentum

$$\mathcal{P}^\alpha = \frac{\partial \mathcal{L}}{\partial \dot{Q}_\alpha} = \mathcal{A}^{\beta\nu,\alpha 0} Q_{\beta,\nu}$$

In the identity (1) the coefficients of each order of time derivative of ϵ^μ must vanish identically. Focusing on the second time derivative term we deduce from $\delta\mathcal{L} = \mathcal{P}^\mu c^0_{\mu r}\ddot{\epsilon}^r + \cdots \equiv 0$ that

$$\mathcal{P}^\mu c^0_{\mu r} \equiv 0$$

These are primary constraints.

$$p_N = \frac{\partial L}{\partial \dot{N}} \equiv 0.$$

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Primary constraints give null vectors of the Legendre matrix $\frac{\partial^2 \mathcal{L}}{\partial \dot{Q}_\alpha \partial \dot{Q}_\alpha}$,

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Legendre matrix is

$$\begin{pmatrix} N^{-1}\eta_{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix}$$

with null vector

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

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The Hamiltonian

Since

$$\mathcal{P}^\alpha = \mathcal{A}^{\alpha 0,\mu 0}\dot{Q}_\mu + \cdots , \quad (2)$$

the velocities are not fixed uniquely in terms of the momenta. Rather,

$$\dot{Q}_\mu = \frac{\partial\, {}^0\mathcal{H}}{\partial \mathcal{P}^\mu} + \lambda^r c^0_{\mu r}$$

$$= \frac{\partial \left({}^0\mathcal{H} + \lambda^r \mathcal{P}^\nu c^0_{\nu r}\right)}{\partial \mathcal{P}^\mu} = \frac{\partial\, \mathcal{H}}{\partial \mathcal{P}^\mu},$$

where the λ^r are arbitrary spacetime functions.

$$H_D = \frac{N}{2}\left(p^2 + m^2 c^2\right) + \lambda p_N$$

and in particular

$$\dot{N} = \lambda$$

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The generator of gauge transformations

Rosenfeld proved that active gauge variations of Q and $\mathcal{P}$ generated by

$$\overline{\mathcal{M}} := \int d^3x \, \mathcal{P}^\alpha \delta Q_\alpha$$

$$-\mathcal{H}\delta x^0 - \mathcal{P}^\alpha Q_{\alpha,a}\delta x^a.$$

$$G = p_N N\dot{\epsilon}$$

$$+\left(\frac{N}{2}\left(p^2 + m^2c^2\right) + \lambda p_N\right)\epsilon$$

So

$$\bar{\delta} q^\mu = N p^\mu \epsilon, \; \bar{\delta} p^\mu = 0,$$

and

$$\bar{\delta} N = \lambda\epsilon + N\dot{\epsilon}$$

Note: λ spoils the group property

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Gauge generator is constant in time

Rosenfeld proved that $\frac{d\overline{\mathcal{M}}}{dt} = 0$. Consequently the coefficients of the time derivatives of ϵ must vanish, and that generally

$$\overline{\mathcal{M}} = \int d^3x \left(\frac{d\epsilon^r}{dt} p^\mu c^0_{\mu r} - \epsilon^r \frac{d}{dt} \left(p^\mu c^0_{\mu r} \right) \right).$$

In other words, the gauge generator vanishes, and secondary constraints arise.

Primary constraint

$$p_N = 0$$

Secondary constraint

$$\left(p^2 + m^2 c^2 \right) = 0.$$

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Rosenfeld and Dirac

Dirac to Rosenfeld, 4/26/31: "Many thanks for sending a copy of your paper on radiation theory, which I have read with great interest." (Niels Bohr Archive)

Rosenfeld to Dirac, 4/30/32: "I enclose a note about your new theory, which is clearly not at all meant um zu kritisieren but nur um zu lernen." (Churchill College Archive)

Rosenfeld publishes demonstration of equivalence of Heisenberg-Pauli and Dirac many-body theory in 1932 - submitted May 2.


St John's College,
Cambridge.
6-5-32.

Dear Rosenfeld,

Thank you very much for the paper you sent me. I found it very interesting. The connection which you give between my new theory and the Heisenberg-Pauli theory is, of course, quite general and holds for any kind of field (not merely the Maxwell kind) in any number of dimensions. This is a very satisfactory state of affairs.

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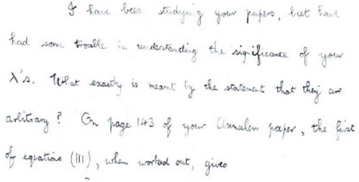
Rosenfeld to Dirac, (5/10/1932) " As to the doubtful sentence of Heisenberg-Pauli, which you are right in not understanding, I would suggest to you to examine the general invariance proof which I give in my paper of the Annalen der Physik, 5, 113, 1930. (I sent you reprints of both)." (Niels Bohr Archive)


I have been studying your papers, but have had some trouble in understanding the significance of your λ's. What exactly is meant by the statement that they are arbitrary? On page 143 of your Annalen paper, the first of equations (III), when worked out, gives

Dirac to Rosenfeld, (5/16/1932) " I have been studying your papers, but have had some trouble in understanding the significance of your λ's. What exactly is meant by the statement that they are arbitrary?" (Niels Bohr Archive)

Rosenfeld to Dirac, (5/21/1932) " As to the λ's, they enter as arbitrary or undetermined coefficients (depending on coordinates) in the general expression of the in terms of the Q's and P's. In equation (111) the hamiltonian should be the same as that of Heisenberg-Pauli (as stated there), so that the substitution of the Ps in terms of the in them will lead to identities, and this implies no restriction for λ. (Churchill College Archive)

Impact of Rosenfeld's work

Pauli to O. Klein, (1/25/1955) "I would like to bring to your attention the work by Rosenfeld in 1930. He was known here at the time as the 'man who quantised the Vierbein (sounds like the title of a Grimms fairy tale doesnt it?) See part II of his work where the Vierbein appears. Much importance was given at that time to the identities among the p's and q's (that is the canonically conjugate fields) that arise from the existance of the group of general coordinate transformations. I still remember that I was not happy with every aspect of his work since he had to introduce certain additional assumptions that no one was satisfied with."

Remark by Dirac

"Well, I think I might answer you in much the same way that I wrote that I felt it had probably been done before, but it was less trouble to me to present it as something new than to search for a reference. A good deal of my work was like that. It happened rather often that there was something which I thought had been done before, but it seemed a great nuisance to look through all the references to try to find it, and if it doesn't take much trouble to publish it, one can publish it again without claiming either that it is new or that it has been done before. (AHQM, 5/10/1963, p 15)"

Bergmann chronology

PHYSICAL REVIEW VOLUME 75, NUMBER 4 FEBRUARY 15, 1949

Non-Linear Field Theories

PETER G. BERGMANN
Department of Physics, Syracuse University, Syracuse, New York

(Received June 8, 1948)

This is the first paper in a program concerned with the quantization of field theories which are covariant with respect to general coordinate transformations, like the general theory of relativity. All these theories share the property that the existence and form of the equations of motion is a direct consequence of the covariant character of the equations. It is hoped that in the quantization of theories of this type some of the divergences which are ordinarily encountered in quantum field theories can be avoided. The present paper lays the classical foundation for this program: It examines the formal properties of covariant field equations, derives the form of the conservation laws, the form of the equations of motion, and the properties of the canonical momentum components which can be introduced.

REVIEWS OF MODERN PHYSICS VOLUME 21, NUMBER 3 JULY, 1949

Non-Linear Field Theories II. Canonical Equations and Quantization*

PETER G. BERGMANN AND JOHANNA H. M. BRUNINGS
Department of Physics, Syracuse University, Syracuse, New York

PHYSICAL REVIEW VOLUME 80, NUMBER 1 OCTOBER 1, 1950

The Hamiltonian of the General Theory of Relativity with Electromagnetic Field*

PETER G. BERGMANN, ROBERT PENFIELD, RALPH SCHILLER, AND HENRY ZATZKIS
Department of Physics, Syracuse University, Syracuse, New York

(Received April 24, 1950)

In this paper we have given a specific example of a Hamiltonian of a non-linear field theory, a Hamiltonian density completely free of time derivatives. In accordance with the general theory developed previously, this Hamiltonian is one of the constraints between the canonical variables and, therefore, vanishes everywhere. To obtain this function, we have developed methods that will also permit the construction of Hamiltonian densities in any field theory in which the Lagrangian density is quadratic in the first derivatives. Our Hamiltonian differs from the one obtained by Schild and Pirani in that they use Dirac's method to derive a Hamiltonian that is invariant but contains velocities, so that their canonical field equations cannot be solved with respect to the time derivatives of all canonical variables. In our formalism, the canonical equations contain no time derivatives on the right-hand sides, but the adoption of a particular Hamiltonian is equivalent to the adoption of a particular coordinate condition and gauge condition. However, once we have obtained any one Hamiltonian density, we can readily obtain any other one (and thus go over to arbitrary coordinate and gauge conditions) by combination with the other constraints of the theory in question.

Hamiltonian constraint is obtained through series of linear transformations that render trivial null vector for Legendre matrix. Second explicit gravitational Hamiltonian (after Pirani and Schild). These first three papers preceeded the discovery of Rosenfelds work by Bergmann's student, J. L. Anderson. All subsequent works of the Bergmann school cited Rosenfeld.

Dirac chronology

- Paul Dirac presents lectures on generalized Hamiltonian dynamics in Vancouver, August 1949
  - Motivation is preservation of Poincaré covariance through parametrization of flat spacetime
  - Alfred Schild and Felix Pirani point out to Dirac applicability to general relativity
- Dirac lectures published in Canadian Journal of Mathematics in 1950 and 1951
- Pirani and Schild submit On the quantization of Einsteins gravitational field equations February 1950
  - Dirac, Bergmann and Brunings (1950) cited
  - First published explicit gravitational Hamiltonian (with note added in press that Bergmann group has obtained same result "using methods quite different from ours")

Dirac's breakthrough

- Dirac, The theory of gravitation in Hamiltonian form, Proc. Roy. Soc. A246, 327 (1958)
  - Time derivatives of temporal components of the metric are eliminated from the Lagrangian through the subtraction of a total time derivative and a spatial divergence
  - g^{0a} are abandoned as canonical variables. Bergmann does likewise.

Arnowitt, Deser, and Misner

- ADM derive Dirac Hamiltonian in a first order Palatini variation. First to employ lapse and shift variables. (R. Arnowitt, S. Deser, C. Misner, Canonical variables for general relativity, Phys. Rev. 117, 1597 (1960))

Bergmann and Dirac

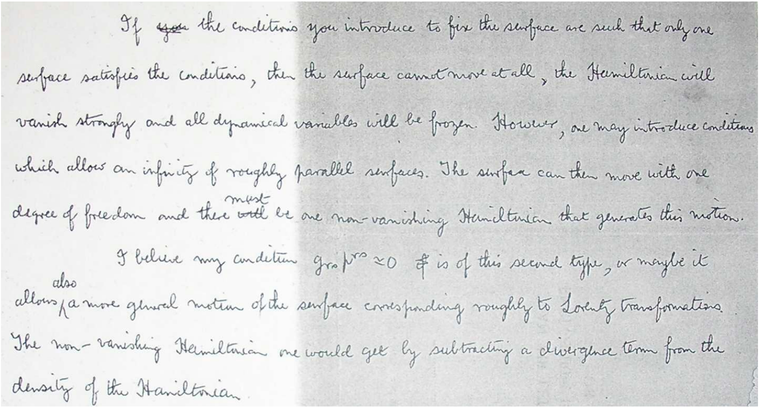
Dear Professor Dirac:

I have just studied your paper that appeared in the May 1 issue of the Physical Review. I am writing you, first to ask you for a reprint when they are available, but I should also like to make a few comments.

(1) The objections that Professor Lichnerowicz and I raised at the end of your lecture at Royaumont, whether or not they were valid then, certainly do not apply to the work that you have published here. Regardless of the motive of introducing the metric g_{rs} on the initial hypersurface, the canonical transformation that you first published a year ago to simplify and kill the primary constraints, is both legimate and sucessful. At this stage the total number of canonical field variables is reduced from twenty to twelve.

Figure: Excerpt of letter from Bergmann to Dirac dated October 9, 1959

(3) When I discussed your paper at a Stevens conference yesterday, two more questions arose, which I should like to submit to you: To me it appeared that because you use the Hamiltonian constraint H_L to eliminate one of the non-substantive field variables, κ, in the final formulation of the theory your Hamiltonian vanishes strongly, and hence all the final field variables, i.e. $\tilde{e}^{rs}$, $\tilde{f}^{rs}$, are "frozen" (constants of the motion). I should not consider that as a source of embarrassment, but Jim Anderson says that in talking to you he found that you now look at the situation a bit differently. Could you enlighten me? If you have no objection, I should communicate your reply to Anderson and a few other participants in the discussion.



If ~~you~~ the conditions you introduce to fix the surface are such that only one surface satisfies the conditions, then the surface cannot move at all, the Hamiltonian will vanish strongly and all dynamical variables will be frozen. However, one may introduce conditions which allow an infinity of roughly parallel surfaces. The surface can then move with one degree of freedom and there ~~will~~ must be one non-vanishing Hamiltonian that generates this motion.

I believe my condition $g_{r0} p^{rs} \approx 0$ is of this second type, or maybe it also allows a more general motion of the surface corresponding roughly to Lorentz transformations. The non-vanishing Hamiltonian one would get by subtracting a divergence term from the density of the Hamiltonian.

Figure: Excerpt of response from Dirac to Bergmann, dated November 11, 1959

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Pons, Salisbury, Shepley and Sundermeyer Formalism

- Pons, Salisbury, and Sundermeyer, "Revisiting observables in generally covariant theories in the light of gauge fixing methods", *PRD* 80, 084015,1-23 (2009)
- Pons and Salisbury, "The issue of time in generally covariant theories and the Komar-Bergmann approach to observables in general relativity", *PRD* 71, 12402 (2005)
- Pons, Salisbury, and Shepley, "Gauge Transformations in the Lagrangian and Hamiltonian Formalisms of Generally Covariant Theories", *PRD* 55, 658-668 (1997)

Projectable Legendre transformations

Projectable infinitesimal general coordinate transformations

$$\epsilon^\mu(x, \phi(x)) = n^\mu(x)\xi^0 + \delta^\mu_a \xi^a$$

with $n^\mu = (N^{-1}, -N^{-1}N^a)$, where N is the lapse and N^a the shift, and ξ^μ are arbitrary infinitesimal functions of the coordinates as well as of the fields ϕ other than the lapse and shift.

$$\epsilon(t) = N^{-1}\xi(t)$$

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Generator of gauge transformations

$$G_\xi(t) = (\mathcal{H}_\mu + N^\rho C^\nu_{\mu\rho} P_\nu)\xi^\mu + P_\mu \dot{\xi}^\mu \,.$$

The $C^\nu_{\mu\rho}$ are the structure functions resulting form the algebra of the Hamiltonian $\mathcal{H}_0$ and momentum $\mathcal{H}_a$ constraints under the Poisson bracket. Note that time dependent canonical gauge transformations alter the functions λ^μ.

$$G = \xi H + p_N \dot{\xi}.$$

$$\bar{\delta} q^\mu = p^\mu \xi = \dot{q}^\mu \epsilon,$$

$$\bar{\delta} N = \dot{\xi} = N\dot{\epsilon} + \dot{N}\epsilon.$$

All spacelike time foliations are accessible and equivalent!

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Gauge fixing through intrinsic coordinates

As a first step towards the explicit form of the functional invariants impose an intrinsic coordinate-dependent gauge condition,
$\chi^{(1)\mu} := x^\mu - X^\mu(x) = 0$,
where the X^μ are spacetime scalar functions of the canonical fields.
Preservation of the gauge conditions under temporal evolution leads to additional constraints
$\chi^{(2)\mu} := \delta^\mu_0 - \mathcal{A}^\mu{}_\nu N^\nu \approx 0$,
where $\mathcal{A}^\mu{}_\rho := \{X^\mu, \mathcal{H}_\rho\}$.

$$\chi^1 = t - q^0 = 0.$$

$$\chi^2 = 1 - \frac{Np^0}{c} = 0$$

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Transformation by finite gauge transformation

It is possible through an appropriate rescaling of the constraints to solve for the finite descriptor of the gauge transformation to the gauge-fixed position. This transformation is then undertaken on all remaining variables. Result is

$$\mathcal{I}_\Phi \approx \Phi + \chi^{(1)\mu}\{\Phi, \overline{\mathcal{H}}_\mu\}$$

$$+\frac{1}{2!}\chi^{(1)\mu}\chi^{(1)\nu}\{\{\Phi, \overline{\mathcal{H}}_\mu\}, \overline{\mathcal{H}}_\nu\}+\cdots$$

$$\overline{H} = \frac{c}{p^0}H,$$

$$\begin{aligned} \mathcal{I}_{q^i} &= q^i + \chi^1\{q^i, \overline{H}\} \\ &= q^i + (t - \frac{q^0}{c})\frac{cp^i}{p^0} \\ &= (q^i - \frac{p^i q^0}{p^0}) + \frac{cp^i}{p^0}t, \\ \mathcal{I}_{p_\mu} &= p_\mu, \\ \mathcal{I}_N &= c/p^0. \end{aligned} \tag{3}$$

Transformation by finite gauge transformation

It is possible through an appropriate rescaling of the constraints to solve for the finite descriptor of the gauge transformation to the gauge-fixed position. This transformation is then undertaken on all remaining variables. Result is

$$\mathcal{I}_\Phi \approx \Phi + \chi^{(1)\mu}\{\Phi, \overline{\mathcal{H}}_\mu\} + \frac{1}{2!}\chi^{(1)\mu}\chi^{(1)\nu}\{\{\Phi, \overline{\mathcal{H}}_\mu\}, \overline{\mathcal{H}}_\nu\} + \cdots$$

$$\overline{H} = \frac{c}{p^0} H,$$

$$\begin{aligned} \mathcal{I}_{q^i} &= q^i + \chi^1\{q^i, \overline{H}\} \\ &= q^i + (t - \frac{q^0}{c})\frac{cp^i}{p^0} \\ &= (q^i - \frac{p^i q^0}{p^0}) + \frac{cp^i}{p^0} t, \\ \mathcal{I}_{p_\mu} &= p_\mu, \\ \mathcal{I}_N &= c/p^0. \end{aligned} \tag{3}$$

Evolving constants and equations of motion

- The coefficients of powers of t are invariant under arbitrary coordinate transformations (and necessarily therefore constants of the motion).
  - We have a group theoretical foundation for Carlo Rovelli's "evolving constants": Rovelli, *PRD* 42, 2638 (1990); 43, 442 (1991)
- The time dependence is naturally what one expects of the gauge-fixed solution.
- Similar expansions exist in the intrinsic spatial coordinates.

Dirac brackets, observables, and transformation to intrinsic coordinates

- Every variable has an associated invariant (including the lapse and shift)
- We have proven that the Poisson bracket algebra of the invariants is the invariant associated with the Dirac bracket

Quantization of the free relativistic particle 1

Equations of motion are

$$\frac{d}{d\,t}\mathcal{I}_{q^i} = \mathcal{I}_{\{q^i,\overline{H}\}} \approx \frac{c}{p^0}\mathcal{I}_{\{q^i,H\}} = c\frac{\mathcal{I}_{p^i}}{\mathcal{I}_{p^0}},$$

$$\frac{d}{d\,t}\mathcal{I}_{p^\mu} = \mathcal{I}_{\{p^\mu,\overline{H}\}} = 0,$$

and

$$\frac{d}{d\,t}\mathcal{I}_{N} = \mathcal{I}_{\{N,\overline{H}\}} = 0.$$

Quantization of the free relativistic particle 2

Required operator algebra is

$$\left[\mathcal{I}_{q^i}, \mathcal{I}_{q^j}\right] = 0,$$

$$\left[\mathcal{I}_{q^i}, \mathcal{I}_{p^j}\right] = i\hbar\delta^{ij},$$

$$\left[\mathcal{I}_{q^i}, \mathcal{I}_{p^0}\right] = i\hbar\frac{\mathcal{I}_{p^i}}{\mathcal{I}_{p^0}},$$

Work in a momentum representation with $\mathcal{I}_{p^0} = (\mathcal{I}_{\vec{p}^2} + m^2c^2)^{1/2}$ and Hamiltonian $H = \mathcal{I}_{p^0}c$!