Léon Rosenfeld and Noether Symmetry Generators

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Overview

1. INTRODUCTION

Introduction

Focus of this talk:

- Major advances in constrained Hamiltonian dynamics made by Léon Rosenfeld in his 1930 paper

Questions to be addressed:

- What were the circumstances that led to Rosenfeld's work?
- How did he exploit the Noether generator in demonstrating not only its full constraint content but also its status as phase space generator of all local symmetries?
- Why have Rosenfeld's achievements been largely unrecognized?

2. BACKGROUND TO ROSENFELD'S 1930 PAPER

"Zur Quantelung der Wellenfelder", *Annalen der Physik* **397**, 113 (1930)

Education and early professional career

- Born 1904 in Charleroi, Belgium
- Completed graduate studies in Paris under the supervision of Louis de Broglie and Théophile de Donder, 1926
- Assistant to Max Born in Göttingen, 1928
- Sought research fellowship, with Einstein's support, to work with Einstein "on the relations between quantum mechanics and relativity"
- Fellowship not awarded and Rosenfeld was invited by Pauli to Zurich, 1929

Pauli's problem

Pauli was uncomfortable with the manner in which he and Heisenberg had treated the $U(1)$ gauge symmetry in their foundational papers in quantum electromagnetic field theory, *Zeitschrift für Physik*, **56**,1 (1929) and **59**, 168 (1930).

Rosenfeld quote, 1963: "I got provoked by Pauli to tackle this problem of the quantization of gravitation and the gravitation effects of light quanta"

Rosenfeld's response

Autobiographical note, 1972: "participated in the elaboration of the theory of quantum electrodynamics just started by Pauli and Heisenberg, and he pursued these studies during the following decade; his main contributions being a general method of representation of quantized fields taking explicit account of the symmetry properties of these fields, a general method for constructing the energy-momentum tensor of any field, a discussion of the implications of quantization for the gravitational field ..."

Rosenfeld's Einstein-Maxwell-Dirac model

Rosenfeld's Einstein-Maxwell-Dirac Lagrangian

$$\mathcal{L} = \frac{1}{2\kappa}(-g)^{\frac{1}{2}} E^{\mu}_{I} E^{\nu}_{J} \left(\omega_{\mu}{}^{I}{}_{L} \omega_{\nu}{}^{LJ} - \omega_{\nu}{}^{I}{}_{L} \omega_{\mu}{}^{LJ} \right)$$

$$+\Re \left\{ (-g)^{1/2} \left[\frac{1}{2} i\bar{\psi}\gamma^{\mu} \left(\overrightarrow{\partial}_{\mu} - i\frac{e}{\hbar c} A_{\mu} + \Omega_{\mu} \right) \psi - m\bar{\psi}\psi \right] \right\} + \mathcal{L}_{em}$$

$$:= \mathcal{L}_{g} + \mathcal{L}_{s} + \mathcal{L}_{em}$$

with tetrads E^{μ}_{I}, electromagnetic potential A_{μ}, Ricci rotation coefficients $\omega_{\nu}{}^{LJ}$, Dirac spinors ψ, and spinor connection $\Omega_{\mu} := \frac{1}{4}\Gamma^{I}\Gamma^{J}\omega_{\mu IJ}$. The Γ^{I} are the Dirac gamma matrices.

3. SYMMETRIES AND FUNDAMENTAL IDENTITIES

Lagrangian symmetry variations

Each of the Lagrangian's transform as scalar densities of weight one under infinitesimal general coordinate transformations

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x),$$

i.e.

$$\delta_C \mathcal{L} = -\mathcal{L}\epsilon^{\mu}_{,\mu}. \tag{1}$$

We also have invariance under $U(1)$ transformations with descriptor ϵ. However, the gravitational Lagrangian yields a total divergence under the local Lorentz transformations

$$\delta_L e_{\mu M} = \epsilon_{MI}\xi^{IJ} e_{\mu J},$$

i.e.,

$$\delta_L \mathcal{L}_g = -\frac{1}{\kappa}\left[\left(E^{\mu}_{I} E^{\nu}_{J} (-g)^{\frac{1}{2}}\right)_{,\nu} \epsilon^{IJ}\right]_{,\mu}. \tag{2}$$

The fundamental identity

Summarizing, we have the fundamental identity (representing the generic field variable by Q_α),

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial Q_\alpha}\delta Q_\alpha + \frac{\partial\mathcal{L}}{\partial Q_{\alpha,\mu}}\delta\left(Q_{\alpha,\mu}\right) \\ &\equiv -\mathcal{L}\epsilon^{\mu}_{,\mu} - \frac{1}{\kappa}\left[\left(E^{\mu}_{I}E^{\nu}_{J}(-g)^{\frac{1}{2}}\right)_{,\nu}\epsilon^{IJ}\right]_{,\mu} \end{aligned} \tag{3}$$

Historical note: the Klein-Noether identities

Felix Klein was the first in 1918 to systematically deduce consequences of these identities for the scalar density case. The extension to variations that yield additional divergences has been attributed to Klein's assistant, Bessel-Hagen (who himself seemed to suggest that he learned of this extension from Emmy Noether). Professor Trautman in 1967 was one of the first to explicitly associate the identities and corresponding conservation laws with Noether-Bessel-Hagen. Rosenfeld seems to have learned of Klein through Pauli. It was Klein who critiqued Pauli's 1921 Encyclopedia article on relativity theory.

4. PRIMARY CONSTRAINTS AND THE CONSTRUCTION OF THE HAMILTONIAN

Primary constraints

The coefficients of the time derivatives of the arbitrary descriptors in the fundamental identity (3)

$$\frac{\partial \mathcal{L}}{\partial \dot{e}_{0I}} = p^{0I} =: \phi^I = 0, \quad \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = p^0 =: \phi = 0,$$

and

$$p^{\mu}_{[I} e_{J]\mu} + p_{\psi} \frac{1}{4} \Gamma_{[I} \Gamma_{J]} \psi + \frac{1}{4} \bar{\psi} \Gamma_{[I} \Gamma_{J]} p_{\bar{\psi}} + \frac{1}{\kappa} \left(\left((-g)^{1/2} E^0_{[I} E^a_{J]} \right)_{,a} \right) =: \phi_{IJ} = 0.$$

Singular Lagrangian and indeterminanacy of velocities

Related to the existence of primary constraints is the singular nature of the Legendre matrix $\frac{\partial^2 \mathcal{L}}{\partial \dot{Q}_\alpha \dot{Q}_\beta}$.

Rosenfeld nevertheless was able to find a general solution for the velocities that involved arbitrary spacetime functions,

$$\dot{Q}_\alpha = \dot{Q}^0_\alpha(Q, p) + \lambda^I c_{\alpha I} + \lambda_{[IJ]} c^{IJ}_\alpha + \lambda c_\alpha,$$

where the $c's$ are null vectors of the Legendre matrix, the $\lambda's$ are arbitrary functions, and Rosenfeld could construct explicitly the special solutions $\dot{Q}^0$.

The constrained Hamiltonian

Rosenfeld then showed that these general solutions could then be substituted into $p^\alpha \dot{Q}_\alpha - \mathcal{L}(Q, \dot{Q})$ to obtain a Hamiltonian that generated the correct Lagrangian equations of motion.

He found that

$$\mathcal{H} = \mathcal{H}_0 + \lambda_I \phi^I + \lambda^{IJ} \phi_{IJ} + \lambda c_\alpha,$$

and

$$\mathcal{H}_0(Q, p) = p^\alpha \dot{Q}^0_\alpha(Q, p) - \mathcal{L}(Q, \dot{Q}^0_\alpha(Q, p)).$$

5. NOETHER, KLEIN AND ROSENFELD'S INFINITESIMAL SYMMETRY GENERATORS

The fundamental identity rewritten

The fundamental identity (3) can equivalently be rewritten as

$$0 \equiv \frac{\delta \mathcal{L}}{\delta Q_\alpha}\delta^* Q_\alpha + \left[\frac{\partial \mathcal{L}}{\partial Q_{\alpha,\mu}}\delta Q_\alpha - \frac{\partial \mathcal{L}}{\partial Q_{\alpha,\mu}} Q_{\alpha,\nu}\xi^\nu + \mathcal{L}\xi^\mu\right]_{,\mu}$$
$$+ \quad \left[\frac{1}{\kappa}\left(E^\mu_I E^\nu_J (-g)^{\frac{1}{2}}\right)_{,\nu}\epsilon^{IJ}\right]_{,\mu}, \qquad (4)$$

where $\frac{\delta \mathcal{L}}{\delta Q_\alpha} = 0$ are the Euler-Lagrange equations.

Aside on Lie derivatives

Following Noether, Rosenfeld defines

$$\delta^* Q_\alpha(x) := Q'_\alpha(x) - Q_\alpha(x),$$

which as Professor Trautman pointed out in 2008, was one year prior to Ślebodziński's introduction of what van Dantzig later dubbed minus the "Lie derivative". Bergmann and his collaborators, beginning in 1949, represented the variation by $\bar{\delta}$. This is actually the notation that was employed by Noether in 1918.

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The conserved current and vanishing Noether charge

Rosenfeld recognized that this identity (4) yielded a conserved current

$$J^\mu = \frac{\partial \mathcal{L}}{\partial Q_{\alpha,\mu}} \delta Q_\alpha - \frac{\partial \mathcal{L}}{\partial Q_{\alpha,\mu}} Q_{\alpha,\nu} \epsilon^\nu + \mathcal{L} \epsilon^\mu + \frac{1}{\kappa} \left(E^\mu_I E^\nu_J (-g)^{\frac{1}{2}} \right)_{,\nu} \epsilon^{IJ},$$

and a vanishing conserved charge

$$M := \int d^3x \left[p^\alpha \delta Q_\alpha - p^\alpha Q_{\alpha,\nu} \epsilon^\nu + \mathcal{L} \epsilon^0 + \frac{1}{\kappa} \left(E^0_I E^a_J (-g)^{\frac{1}{2}} \right)_{,a} \epsilon^{IJ} \right]$$
$$= \int d^3x \left[p^\alpha \delta Q_\alpha - \mathcal{H} \epsilon^0 - \mathcal{H}_a \epsilon^a + \frac{1}{\kappa} \left(E^0_I E^a_J (-g)^{\frac{1}{2}} \right)_{,a} \epsilon^{IJ} \right]. \qquad (5)$$

The phase space generator of infinitesimal symmetry transformations

Rosenfeld was able prove that M generated the correct infinitesimal variations of all of the phase space variables.

He proved that the correct variations of p^{α} were generated by making use of the fundamental identity (3)!

Thus, in addition to $U(1)$ and local Lorentz transformations, he also had the correct generator of infinitesimal general coordinate transformations.

The Rosenfeld-Noether generator and primary constraints

The Rosenfeld-Noether generator in terms of the primary constraints is

$$
\begin{aligned}
M &= \int d^3x \left[-\phi^I e_{0I}\dot{\epsilon}^0 - \phi^I e_{aI}\dot{\epsilon}^a - \phi\dot{\epsilon}\right. \\
&+ \left(\left(p^{aI} e_{0I}\right)_{,a} + (p^a A_0)_{,a} - \mathcal{H}\right)\epsilon^0 \\
&+ \left(\left(p^{aI} e_{bI}\right)_{,a} - \mathcal{H}_b\right)\epsilon^b \\
&+ \left(-p^a_{,a} + i\frac{e}{\hbar c}p_\psi \psi - i\frac{e}{\hbar c}p_{\psi^\dagger}\psi^\dagger\right)\epsilon \\
&+ \left.\phi_{IJ}\epsilon^{IJ}\right] = 0. \qquad (6)
\end{aligned}
$$

Secondary constraints

Rosenfeld observed that since M vanishes for arbitrarily time-dependent descriptors, each line in the previous expression must separately vanish. Thus Rosenfeld was able to derive secondary constraints even without explicit knowledge of the Hamiltonian!

Furthermore, He observed that the coefficients of ϵ^μ are the four Einstein equations that do not involve accelerations.

6. FROM INFINITESIMAL TO FINITE CANONICAL SYMMETRY TRANSFORMATIONS

The problem with coordinate time transformations

Rosenfeld did observe (at least implicitly) that it was not possible to implement finite coordinate transformations $x^0(x)$ using this generator. This posed a serious obstacle to Bergmann's quantization program in the 1950's, as he recollected in 1979:

"During the early Fifties those of us interested in a Hamiltonian formulation of general relativity were frustrated by a recognition that no possible canonical transformations of the field variables could mirror four-dimensional coordinate transformations and their commutators, not even at the infinitesimal level. That is because (infinitesimal or finite) canonical transformations deal with the dynamical variables on a three-dimensional hypersurface, a Cauchy surface, and the commutator of two such infinitesimal transformations must be an infinitesimal transformation of the same kind."

The problem reflected in the Rosenfeld-Noether generator

In working out the variations generated by the Rosenfeld-Noether charge one most make explicit use of the Hamiltonian equations of motion and replace the arbitrary functions λ by corresponding field derivatives. But there is no canonical means of updating these assignments for subsequent infinitesimal transformations.

Dirac's breakthrough and the Bergmann Komar group

Although Dirac never concerned himself with the question whether the full diffeomorphism group could be realized as a canonical transformation group, he is the one who in 1958 unintentionally invented the framework in which this goal could be achieved. The key was the decomposition of infinitesimal coordinate transformations which were either tangent to a given foliation of spacetime into fixed time slices, or perpendicular to the foliation.

In 1972 Bergmann and Komar subsequently gave a group-theoretical interpretation of this decomposition, pointing out that the relevant group was a phase space transformation group that possessed a compulsory dependence on the spacetime metric.

The Legendre projectability requirement

We have an mathematical justification for the Dirac decomposition. It is required in order that configuration-velocity variations be projectable under the Legendre transformation to phase space.

The generator of infinitesimal transformations

Confining our attention to general coordinate transformations the Noether generator density is

$$\mathcal{G}_\epsilon(t) = P_\mu \dot{\epsilon}^\mu + (\mathcal{H}_\mu + \int d^3x' \int d^3x'' N^{\rho'} C^{\nu''}_{\mu\rho'} P_{\nu''})\epsilon^\mu .$$

where

$$\{\mathcal{H}_\mu(x), \mathcal{H}_\rho(x')\} = C^{\nu''}_{\mu\rho'}\,[g_{ab}]\,\mathcal{H}_{\nu''}$$

Comparison with the Rosenfeld-Noether generator

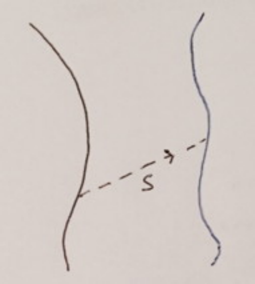
In every example we have checked, this generator agrees with the Rosenfeld-Noether with the simple replacement $\epsilon^\mu = n^\mu \xi^0 + \delta^\mu_a \xi^a$.

Time evolution versus diffeomorphisms

The evolution in time is generated by

$$H = \int d^3x \left(N\mathcal{H}_0 + N^a\mathcal{H}_a + \lambda_\mu P^\mu \right).$$

The finite diffeomorphism generator $\exp\left(s \int d^3x \, \mathcal{G}_\epsilon(t)\right)$ transforms solutions into new solutions.



Enlargement of phase space

Note that the lapse function N and shift N^a must be retained as canonical variables.

Note also that contrary to popular belief, the Hamiltonian formulation does not fix a time foliation. New foliations result in new multipliers λ^μ and new Hamiltonians as a consequence of the time dependence of the Hamiltonian.

7. IMPLICATIONS FOR CANONICAL QUANTUM GRAVITY

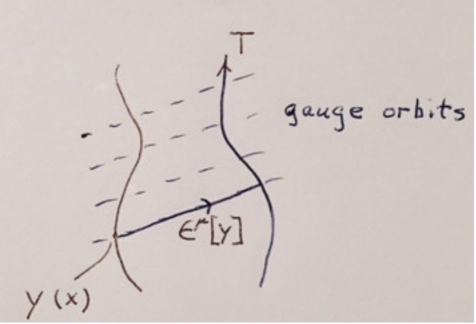
Diffeomorphism invariants constructed with finite group elements

Now that we have the full diffeomorphism group at our disposal, we can employ it to establish correlations between partial variables. One possible implementation, in principle, is to locate temporal and spatial landmarks by referring to curvature even in the vacuum case. There are of course many more possibilities when matter is present. We will employ these landmarks as "intrinsic" coordinates. Such coordinates must be formed from spacetime scalars. Thus we choose $X^\mu[g_{ab}, p^{ab}]$.

In the vacuum case we propose the use of the four Weyl curvature scalars, as originally suggested by Komar in the 1950's. They are quadratic and cubic in the Weyl tensor. Bergmann and Komar showed in 1960 that they are expressible solely in terms of the three metric and its conjugate momenta.

Intrinsic coordinate gauge conditions

We choose intrinsic coordinates through the gauge conditions $x^\mu = X^\mu[g_{ab}, p^{ab}]$. Given any solution trajectory in phase space we can then determine the phase space dependent finite descriptors $\epsilon^\mu[g_{ab}, p^{ab}] := \epsilon^\mu[y]$ that will gauge transform these solutions to those that satisfy the gauge conditions.



The explicit construction of evolving constants of the motion

This construction yields Taylor expansions in the coordinates x^μ - now themselves diffeomorphism invariants. The coefficients in the Taylor expansions are functionals of g_{ab} and p^{ab} that are explicitly diffeomorphism invariants. This applies also to the invariant lapse and shift.

$$\mathcal{I}_\phi = \sum_{n_\mu=0}^{\infty} \frac{1}{n_0!\, n_1!\, n_2!\, n_3!} (x^0)^{n_0} (x^1)^{n_1} (x^2)^{n_2} (x^3)^{n_3}\, \mathcal{C}_{n_0,n_1,n_2,n_3}[g_{ab}, p^{ab}]$$

References I