Full four-dimensional diffeomorphism invariants and their role in quantum theories of gravity

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Overview

1. INTRODUCTION

Introduction

Focus of this talk:

- What are some implications for an eventual quantum theory of gravity of the classical evolution of the metric with respect to temporal and spatial landmarks constructed with the use of Weyl curvature scalars. All variables are diffeomorphism invariants.

Questions to be addressed:

- What is the status of the multiplicity of observer-based evolution in classical general relativity.
- How can the fully relational approach be applied to loop quantum gravity?
- Is there a meaningful quantum generalization to non-commuting intrinsic coordinate algebras in loop quantum gravity?

2. BRIEF HISTORY OF HAMILTONIAN GENERAL COVARIANCE

Rosenfeld's 1930 tetrad gravitational Lagrangian

"Zur Quantelung der Wellenfelder", *Annalen der Physik* **397**, 113 (1930) Translation by Salisbury and Sundermeyer [Rosenfeld, 2017]

$$\mathcal{L} = \frac{1}{2\kappa}(-g)^{\frac{1}{2}} E^\mu_I E^\nu_J \left(\omega_\mu{}^I{}_L \omega_\nu{}^{LJ} - \omega_\nu{}^I{}_L \omega_\mu{}^{LJ}\right)$$
$$+\Re\left\{(-g)^{1/2}\left[\frac{1}{2} i\bar{\psi}\gamma^\mu\left(\overrightarrow{\partial}_\mu + \Omega_\mu\right)\psi - m\bar{\psi}\psi\right]\right\} + \mathcal{L}_{em}$$

Tetrads E^μ_I, Rotation coefficients $\omega_\mu{}^I{}_L$, Fermion field ψ, spinor connection $\Omega_\mu = \frac{1}{4}\gamma^I\gamma^J\omega_{\mu IJ}$.

Rosenfeld's 1930 tetrad Hamiltonian density

Rosenfeld invented a systematic procedure for solving for the velocities $\dot{E}^\mu_I$ in terms of the conjugate momenta given that the Jacobian matrix $\frac{\partial^2 \mathcal{L}}{\partial \dot{E}^\mu_I \dot{E}^\nu_J}$ is singular. Although he did not do this explicitly for this model, the result (see [Salisbury & Sundermeyer, 2017]) is

$$\mathcal{H} = \mathcal{H}_0 \left[g_{ab}, p^{ab}, A_a, p^a, \psi, \psi^\dagger\right] + \lambda_I \mathcal{F}^I + \lambda_{IJ} \mathcal{F}^{[IJ]} + \lambda \mathcal{F}$$

where $\mathcal{F}^I$, $\mathcal{F}^{[IJ]}$ and $\mathcal{F}$ are primary constraints and λ_I, λ_{IJ} and λ are arbitrary spacetime functions.

Preceeding Bergmann and Dirac by twenty years! See [Salisbury, 2009].

Rosenfeld's infinitesimal phase space symmetry generator

Rosenfeld proved that the vanishing Noether charge generated the correct variations of all of the phase space variables under all of the local symmetries. Most importantly for us is that the active variations under the infinitesimal coordinate transformations $x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$ are correct.

His conserved and vanishing generating density is

$$-\mathcal{F}^I e_{0I}\dot{\xi}^0 - \mathcal{F}^I e_{aI}\dot{\xi}^a - \mathcal{F} A_0\dot{\xi}^0 - p^{aI} e_{\nu I}\xi^{\nu}_{,a} - p^a A_{\nu}\xi^{\nu}_{,a} - \mathcal{H} A_0\xi^0 - \mathcal{G}_a\xi^a$$

$$-\mathcal{F}\dot{\xi} + p^a\xi_{,a} + i\frac{e}{\hbar c}p_{\psi}\psi\xi - i\frac{e}{\hbar c}p_{\psi^{\dagger}}\psi^{\dagger}\xi + \mathcal{F}_{[IJ]}\xi^{IJ} = 0$$

Peter Bergmann and Paul Dirac

We leap two decades forward to the contributions to constrained Hamiltonian dynamics of Peter Bergmann and Paul Dirac - beginning in 1949.

Dirac never concerned himself with the phase space realization of the full general covariance group. See his Vancouver lectures, [Dirac, 1950] [Dirac, 1951]

[Bergmann, 1949], later with Jim Anderson [Anderson & Bergmann, 1951] and numerous collaborators including [Goldberg, 1953] did concern themselves with this symmetry. In particular a joint publication with Ralph Schiller [Bergmann & Schiller, 1953] explicitly employed the vanishing Noether charge.

Non-realizability of the diffeomorphism Lie algebra

But there is an obstacle to the realization of finite diffeomorphisms, explicitly recognized by Bergmann. One can see this in the Lie algebra

$$\xi_3^\mu = \xi_{1,\nu}^\mu \xi_2^\nu - \xi_{2,\nu}^\mu \xi_1^\nu = \xi_{1,0}^\mu \xi_2^0 - \xi_{2,0}^\mu \xi_1^0 + \dots$$

Repeated commutators lead to higher and higher order time derivatives.

Dirac's resolution

Dirac's solution was probably inspired by his student Paul Weiss: write the infinitesimal variations as a sum of perpendicular and tangent increments,

$$\xi^\mu = n^\mu \epsilon^0 + \delta^\mu_a \epsilon^a$$

This results in the familiar metric dependent Dirac algebra.

The Bergmann - Komar group

[Bergmann & Komar, 1972] interpreted this algebra as representing a compulsory metric-dependent transformation group

3. GAUGE CONDITIONS AND ASSOCIATED INVARIANTS

The Legendre projectability requirement

We have an mathematical justification for the Dirac decomposition. It is required in order that configuration-velocity variations be projectable under the Legendre transformation to phase space, [Pons *et al.* , 1997]

Ashtekar-Barbero-Immirzi Lagrangian

[Pons *et al.* , 2000], [Pons & Salisbury, 2002] The connection is

$$ {}^{\alpha}A^i_a := \omega^i_a - \alpha^{-1} K^i_a, \;\; {}^{\alpha}A^i_0 := \Omega^i_0 - \alpha^{-1} K^i_a N^a + \alpha T^a_i N_{,a} $$

The Lagrangian is

$$ \mathcal{L}_{ABI} = \alpha^{-1\alpha} A^i_a \dot{\tilde{T}}^a_i + N^a \tilde{\mathcal{H}}_c + \underset{\sim}{N}{}^{\alpha}\tilde{\tilde{\mathcal{H}}}_0 + {}^{\alpha}A^i_0 {}^{\alpha}\tilde{\mathcal{H}}_i , $$

where the secondary constraints are

$$ {}^{\alpha}\tilde{\mathcal{H}}_i := -\alpha {}^{\alpha}D_a \tilde{T}^a_i = 0, \;\; {}^{\alpha}\tilde{\mathcal{H}}_a := \alpha \tilde{T}^{b\alpha}_i F^i_{ba} = 0 $$

and

$$ {}^{\alpha}\tilde{\tilde{\mathcal{H}}}_0 := -\frac{1}{2} \tilde{T}^a_i \tilde{T}^b_j (-\alpha^{2\alpha} F^{ij}_{ab} + (1+\alpha^2)^3 R^{ij}_{ab}) = 0. $$

Legendre-projectable infinitesimal symmetries

It is noteworthy that when gauge symmetries are present in addition to diffeomorphism symmetry, the transformations of some of the additional gauge variables under the change of coordinates

$$x'^{\mu} = x^{\mu} - n^{\mu}\xi^{0} - \delta^{\mu}_{a}\xi^{a},$$

are no longer projectable. See [Pons *et al.* , 2000]. In this case the variation of ${}^{\alpha}A^{i}_{0}$ acquires unprojectable time derivatives of N, N^{a}, and ${}^{\alpha}A^{i}_{0}$. But these can be removed by adding to the variation an $SO(3)$ rotation with descriptor $\left(-n^{\mu\alpha}A^{i}_{\mu} + \alpha N^{-1}T^{bi}N_{,b}\right)$

This is in fact the variation generated by ${}^{\alpha}\tilde{\tilde{\mathcal{H}}}_{0}$.

The complete generator of infinitesimal symmetry transformations

The primary constraints are the variables canonically conjugate to the lapse, shift, and time component of the connection. Combination of primary and secondary constraints, all first class, allows one to construct the full set of gauge generators

$$\mathcal{G}_\xi = P_A \dot{\xi}^A + (\mathcal{H}_A + P_{C''} N^{B'} \mathcal{C}^{C''}_{AB'})\xi^A,$$

where $\mathcal{C}^{C''}_{AB'}$ are the structure functions associated with the Poisson brackets of the secondary constraints and the descriptors ξ are infinitesimal arbitrary functions of spacetime coordinates appearing in the projectable infinitesimal coordinate transformations

$$x'^\mu = x^\mu - n^\mu \xi^0 + \delta^\mu_a \xi^a,$$

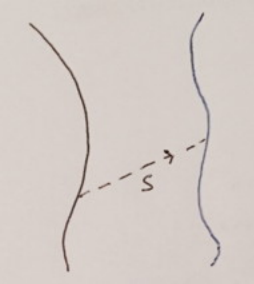
and local $SO(3)$ gauge rotations - generated by $\xi^{i\alpha}\widetilde{\mathcal{H}}_i$.

Time evolution versus diffeomorphisms

The evolution in time is generated by

$$\int d^3x \mathcal{H}_{ABI} = \int d^3x \left(-{}^{\alpha}A_0^i\, {}^{\alpha}\tilde{\mathcal{H}}_i + N^a\, {}^{\alpha}\tilde{\mathcal{H}}_a + \underset{\sim}{N}\, {}^{\alpha}\tilde{\tilde{\mathcal{H}}}_0 + \lambda_A P^A \right).$$

But the finite diffeomorphism generator $\exp\left(s \int d^3x\, \mathcal{G}_{\xi}(t)\right)$ transforms solutions into new solutions.



Enlargement of phase space

Note that the lapse function N, the shift N^a and the nought component of the connection ${}^{\alpha}A_0^i$ must be retained as canonical variables.

Note also that contrary to popular belief, the Hamiltonian formulation does not fix a time foliation. New foliations result in new multipliers λ^A and new Hamiltonians as a consequence of the time dependence of the Hamiltonian.

4. INTRINSIC GRAVITATIONAL HAMILTON-JACOBI APPROACH

An implementation of Rovelli's partial variable program

Now that we have the full diffeomorphism group at our disposal, we can employ it to establish correlations between partial variables. One possible implementation, in principle, is to locate temporal and spatial landmarks by referring to curvature even in the vacuum case. There are of course many more possibilities when matter is present. We will employ these landmarks as "intrinsic" coordinates. Such coordinates must be formed from spacetime scalars. Thus we choose $X^\mu[{}^\alpha A^i_a, \tilde{T}^b_j]$.

In the vacuum case we propose the use of the four Weyl curvature scalars, as originally suggested by [Komar, 1958]. They are quadratic and cubic in the Weyl tensor.
[Bergmann & Komar, 1960] showed that they are expressible solely in terms of the three metric and its conjugate momenta. The same logic demonstrates dependence only on ${}^\alpha A^i_a$ and $\tilde{T}^b_j$

Weyl scalars in the Ashtekar program

In fact, we have for the complex connections that the Newman-Penrose scalars are

$$\psi_{ij} = \frac{1}{4} \underset{\sim}{t}_a^{(i} \epsilon^{abc} F_{bc}^{j)}.$$

(See [Salisbury *et al.* , 1994], summarizing results originally due to [Capovilla *et al.* , 1991]). The Weyl scalars are in turn expressible as $I = \psi_i{}^j \psi_j{}^i$ and $J = \psi_i{}^j \psi_j{}^k \psi_k{}^i$. (See [Penrose & Rindler, 1988])

Proof of principle

Weyl scalars can always serve as coordinates in a local Riemann normal coordinate system. Overlapping patches can then cover the entire manifold.

1. Expand the metric in the neighborhood of any spacetime event to third order. See, for example, [Brewin, 2009]

$$g_{\mu\nu}(x) = g_{\mu\nu} - \frac{1}{3}x^\rho x^\sigma R_{\mu\nu\rho\sigma} - \frac{1}{6}x^\rho x^\sigma x^\kappa \partial_\kappa R_{\mu\nu\rho\sigma} + \mathcal{O}(\epsilon^4)$$

2. Keep only the linear in x^μ contributions to I and J and solve for the x^μ in terms of the I and J.

Non-trivial classical evolution with spatial landmarks

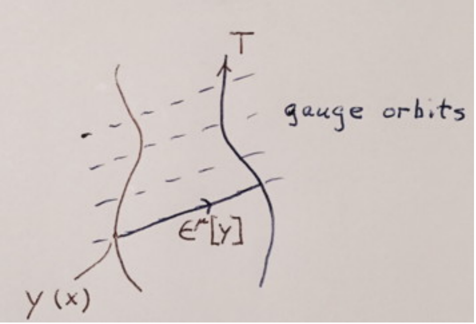
The resulting evolution is with respect to partial variables - in the sense of [Rovelli, 1991].

There is also obvious variation in spatial directions. There is more to space than topology!

We shall see that having selected intrinsic coordinates the evolution is unique with regard to the in principle measureable curvature coordinates. The behavior is insensitive to whatever coordinates one employed before transforming to intrinsic coordinates. But on the other hand, given any initial intrinsic coordinates one can undertake arbitrary changes in these new intrinsic coordinates - yielding physically distinguishable evolutions. Thus we have a paradoxical situation where we are dealing with general coordinate invariants, and yet we can meaningfully arbitrarilly alter the intrinsic coordinate choices. Rovelli has referred to this phenomena as involving evolving constants of the motion.

Intrinsic coordinate gauge conditions

We choose intrinsic coordinates through the gauge conditions $x^\mu = X^\mu[g_{ab}, p^{ab}]$. Given any solution trajectory in phase space we can then determine the phase space dependent finite descriptors $\epsilon^\mu[g_{ab}, p^{ab}] := \epsilon^\mu[y]$ that will gauge transform these solutions to those that satisfy the gauge conditions.



The explicit construction of evolving constants of the motion

This construction yields Taylor expansions in the coordinates x^μ - now themselves diffeomorphism invariants. The coefficients in the Taylor expansions are functionals of g_{ab} and p^{ab} that are explicitly diffeomorphism invariants. This applies also to the invariant lapse and shift.

$$\mathcal{I}_\phi = \sum_{n_\mu=0}^{\infty} \frac{1}{n_0!\, n_1!\, n_2!\, n_3!} (x^0)^{n_0} (x^1)^{n_1} (x^2)^{n_2} (x^3)^{n_3}\, \mathcal{C}_{n_0,n_1,n_2,n_3}[g_{ab}, p^{ab}]$$

Kuchar-inspired canonical transformations

Canonical transformations can in principle be carried out to new canonical variables including X^{μ} and canonical conjugates π_{μ} - but without imposing gauge conditions. The theory in terms of these new variables is still fully diffeomorphism covariant - with corresponding Hamiltonian constraints. Each choice yields a new form for the constraints and a new Wheeler-DeWitt equation with a corresponding "natural" choice of temporal and spatial partial variables - with the scalar constraint now expressed in terms of the X^{μ}.

This "natural" choice is the one that results through the solutions of the Wheeler-DeWitt equation.

Free relativistic particle example

Choose as the intrinsic evolution parameter the proper time. This corresponds to a canonical change of $T = -m\frac{q^0}{p_0}$, and our task is to find the canonical generating function $G(q^0, T)$ such that the simplectic one-form contribution $p_0 dq^0$ becomes

$$PdT + \frac{\partial G}{\partial q^0} dq^0 + \frac{\partial G}{\partial T} dT.$$

Having made the canonical change of variables, we of not yet made a choice of an intrinsic time. The rewritten mass shell constraint still generates arbitrary infinitesimal reparamtereizations of the form $\theta' = \theta - \left(-\dot{q}^2\right)^{-1/2} \xi(\theta)$. This change is in fact generated by the transformed mass shell constraint, with the generator taking the form

$$0 = \xi \left(P + \ln(T)\right) + \frac{1}{2}\left(p^a p_a + m^2\right),$$

with intrinsic Hamiltonian $H = \frac{1}{2}\left(p^a p_a + m^2\right)$

I can now choose the proper time as the intrinsic evolution parameter by making the gauge choice $\theta = T$ and eliminating its momentum conjugate by solving for P. The result is that the simplectic form becomes

$$dS = \left[-\frac{1}{2m} \left(p^a p_a + m^2 \right) + \ln(\theta) \right] d\theta + p_a dq^a .$$

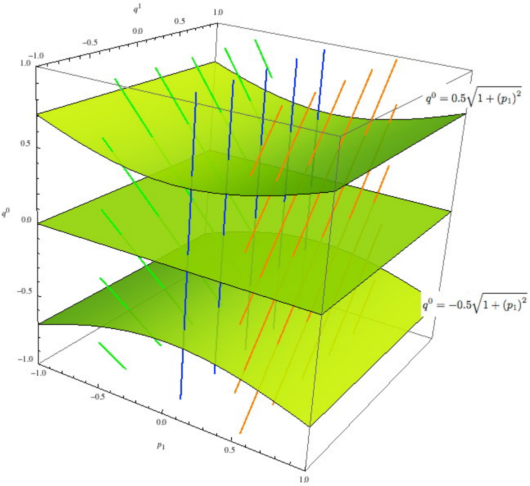


Figure: Proper time slicing in one spatial dimension of free particle gauge orbits, where the proper time values are -0.5, 0, and .5. The particle mass is taken to be one.

Ashtekar - Intrinsic Hamilton-Jacobi Approach

The variation of the Ashtekar-Barbero-Immirzi action about solutions is

$$\int d^3x \left({}^{\alpha}A^i_a \delta \tilde{T}^a_i - \mathcal{H}_{ABI} \delta t - {}^{\alpha}\mathcal{H}_a \delta x^a \right)$$

We seek a canonical change of variables $\left(\tilde{T}^a_i, A^i_a \delta \right) \rightarrow \left(X^\mu, \pi_\nu, g_A, p^A \right)$ such that the non-vanishing contribution to the symplectic one-form becomes

$$\int d^3x \, {}^{\alpha}A^i_a d\tilde{T}^a_i = \int d^3x \left(\pi_\mu dX^\mu + p^A dg_A + \frac{\delta G}{\delta \tilde{T}^a_i} d\tilde{T}^a_i + \frac{\delta G}{\delta g_A} dg_A \right)$$

Ashtekar - Intrinsic Hamilton-Jacobi Approach

The next step is to solve the constraints and the gauge conditions, thereby replacing the canonical variables X^μ by x^μ, and eliminating the conjugates π_μ. The result is an explicitly time dependent Hamiltonian.

5. IMPLICATIONS FOR QUANTUM GRAVITY?

Coordinates matter

Standard canonical approaches to quantum gravity involve either explicitly or implicitly a preferred choice of coordinates. Missing is reference to observers - or equivalently from the perspective of this work a reference to the partial variables with respect to which one is to contemplate observations. In fact, classically the choice is at least as great as the choice of intrinsic Weyl curvature coordinates.

Which Wheeler-DeWitt equation

For each choice of canonical variables X^μ there is a corresponding Wheeler-DeWitt equation.

Setting $x^\mu = X^\mu$ results in a time-dependent Schroedinger equation - and in general non-unitary evolution.

Gravitational entanglement entrop?

Try pursuing canonical quantization in locally flat approximation. Can one imitate the Jacobson argument with respect to a local Rindler frame to get the empty space Einstein equations?

Canonical loop quantum gravity

All approaches invoke a partial gauge fixing - taking $N = N^a = A_0^i = 0$. The general covariance is lost and observer frames are only partially fixed. A unique fixation requires explicit coordinate dependence.

Intrinsic coordinates for loop quantum gravity?

Current project: explore possibility of taking the variables ψ_{ij}, or more precisely the related scalars $\psi_0, \ldots, \psi_4$, the dyad spinor components of the Weyl spinor, where

$$\psi_{11} = \frac{1}{2}\left(-\psi_0 + 2\psi_2 - \psi_4\right), \ \ \psi_{12} = \frac{i}{2}\left(\psi_0 - \psi_4\right), \ \ \psi_{13} = \psi_1 - \psi_3,$$

$$\psi_{22} = \frac{1}{2}\left(\psi_0 + 2\psi_2 + \psi_4\right), \ \ \psi_{23} = -i\left(\psi_1 + \psi_3\right), \ \ \psi_{33} = -2\psi_2$$

as independent variables in the Barbero-Immirzi formalism. Then the isolation of the invariants I and J is algebraically trivial.

Non-commutative geometry?

There is are almost obvious candidates for a non-commutative geometrical approach: Do not bother to find canonical variables X^μ that commute! But then what criteria would would apply to obtain the non-commuting operator choices?

Could a Planck scale uv cutoff be introduced in this manner?

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