Observables in General Relativity

Don Salisbury
Austin College

North Texas Relativity Seminar - September 15, 2005

Plan of Talk

1. Motivation
2. Projectability of diffeomorphism symmetries under Legendre map
3. Diffeomorphism-induced symmetry generators and Hamiltonian
4. Finite symmetry transformations and time evolution
5. Gauge fixing using intrinsic coordinates
6. Time-dependent diffeomorphism invariants
7. Bianchi Type I cosmology
8. Quantum implications
9. Conclusions

1 - Motivation

- Desire to realize 4-D diffeomorphism symmetry in canonical approach to quantum gravity
- Lapse and shift should be quantum operators subject to quantum fluctuations
- We all know intuitively that “frozen time” is nonsense!

Collaborators and references

- "The issue of time in generally covariant theories and the Komar-Bergmann approach to observables in general relativity," (with J Pons) Phys. Rev. D 71 124012 (2005)
- "Reparameterization invariants for anisotropic Bianchi I cosmology with a massless scalar source," (with J. Helpert and A. Schmitz) gr-qc/0503014
- "The gauge group in the Ashtekar-Barbero formulation of canonical gravity,", in Proceedings of the Ninth Marcel Grossmann Meeting, edited by V.G. Gurzadyan, R. T. Jantzen and R. Ruffini, (World Scientific, New Jersey, 2002), 1298 (with J. Pons)
- "The gauge group and the reality conditions in Ashtekar's formulation of general relativity," Phys. Rev. **D62** , 064026 (2000) (with J.M. Pons and L.C. Shepley)
- "The gauge group in the real triad formulation of general relativity," Gen. Rel. Grav. **32**, 1727 (2000) (with J.M. Pons and L.C. Shepley)
- "Gauge transformations in Einstein-Yang-Mills theories," J. Math. Phys. **41**, 5557 (2000) (with J.M. Pons and L.C. Shepley)
- "The realization in phase space of general coordinate transformations," Phys. Rev. D27, 740 (1983) (with K. Sundermeyer)

2 - Legendre projectability of diffeomorphism symmetries

- All generally covariant models have singular Lagrangians

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) = 0$$

- Configuration-velocity functions which vary in direction of null directions are not projectable to phase space

if $\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\gamma^j = 0$, then for $f(q,\dot{q})$ to be projecta

it must satisfy $\gamma^j \frac{\partial f}{\partial \dot{q}^j} = 0$

Proof - we mean by “projectable” that f is the pullback of a function *F(q, p)* on phase space:

$$\gamma^j \frac{\partial f(q,\dot{q})}{\partial \dot{q}^j} = \frac{\partial F(q,p(q,\dot{q}))}{\partial p_k} \gamma^j \frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^j} = 0$$

Cosmology example

$$g_{\mu\nu} = \begin{pmatrix} -N^2 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & a^2 \end{pmatrix}$$

$$L = -\frac{6\dot{a}^2 a}{\kappa N} + \frac{a^3\dot{\phi}^2}{2N}$$

where κ is $8\pi G$ and ϕ is a massless scalar material field

Therefore

$$\frac{\partial^2 L}{\partial \dot{q}\partial \dot{q}}\begin{pmatrix}0\\0\\1\end{pmatrix} = 0 \text{ since} \frac{\partial L}{\partial \dot{q}^3} \equiv \frac{\partial L}{\partial \dot{N}} = 0$$

So projectable functions cannot depend on $\dot{N}$

Austin College 1849

- Consider variations of metric under infinitesimal coordinate transformations

$$x'^{\mu} = x^{\mu} - \varepsilon^{\mu}(x)$$

Contains time derivatives of lapse and shift

$$\delta g_{\mu\nu} = g_{\mu\nu,\alpha}\varepsilon^{\alpha} + g_{\alpha\nu}\varepsilon^{\alpha}_{,\mu} + g_{\mu\alpha}\varepsilon^{\alpha}_{,\nu}$$

where

lapse

shift

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + g_{cd}N^c N^d & g_{ac}N^c \\ g_{bc}N^c & g_{ab} \end{pmatrix}$$

Not projectable

cosmology example

$$t' = t - \varepsilon(t), \text{ so } \delta g_{00} = g_{00,0}\varepsilon + 2g_{00}\dot{\varepsilon} \Rightarrow \delta N = \dot{N}\varepsilon + N\dot{\varepsilon}$$

- Resolution: infinitesimal coordinate transformations must depend in a unique, precise way on the lapse and shift

$$\varepsilon^{\mu}(x) = \delta^{\mu}_{a}\xi^{a}(x) + n^{\mu}(x)\xi^{0}(x)$$

where

$$n^{\mu} = \left(N^{-1}, -N^{-1}N^{a}\right)$$

is the normal to the constant time hypersurface

cosmology example:

$$t' = t - N^{-1}\xi \Rightarrow \delta N = \dot{N}N^{-1}\xi + N\frac{d}{dt}\left(N^{-1}\xi\right) = \dot{\xi}$$

3 - Symmetry Generators and Hamiltonian

Primary constraints

Secondary constraints

$$G[\xi] = \int d^3x \left(\dot{\xi}^{\mu} P_{\mu} + \xi^{\mu} \left(\mathrm{H}_{\mu} + \iint d^3y \, d^3z \, C^{\beta}_{\mu\alpha}(x,y,z) N^{\alpha}(y) P_{\beta}(z) \right) \right)$$

$$G[\xi] = \dot{\xi} p_N + \xi \left(-\frac{3\kappa p_a^2}{2a} + \frac{p_\phi^2}{2a^3} \right)$$

Group structure functions: $\left\{ \mathrm{H}_{\mu}(x), \mathrm{H}_{\nu}(y) \right\}_{PB} = \int d^3z \, C^{\beta}_{\mu\nu}(x,y,z) \mathrm{H}_{\beta}(z)$

Cosmology example

$$G[\xi] = \dot{\xi} p_N + \xi \left(-\frac{\kappa p_a^2}{24a} + \frac{p_\phi^2}{2a^3} \right)$$

Hamiltonian

Functions of dynamical canonical variables

$$H = \int d^3x \left(N^{\mu} \mathrm{H}_{\mu} + \lambda^{\mu} P_{\mu} \right)$$

Arbitrary functions of coordinates

Cosmology example:

$$H = N\left(-\frac{\kappa p_a^2}{24a} + \frac{p_\phi^2}{2a^3} \right) + \lambda p_N$$

4 - Finite Time Evolution and Symmetry Transformations

Finite time evolution operator:

Time ordering

$$\hat{U}(t,0) = \mathrm{T}\exp\left(\int_0^t dt'\{\ \ ,H(t')\}_{PB}\right)$$

$$= 1 + \int_0^t dt_1\{\ \ ,H(t_1)\}_{PB} + \int_0^t dt_1 \int_0^{t_1} dt_2\{\{\ \ ,H(t_1)\}_{PB},H(t_2)\}_{PB} + \ldots$$

Cosmology example:

$$H(t) = N_0\left(-\frac{\kappa p_{a0}^2}{24 a_0} + \frac{p_{\phi 0}^2}{2 a_0^3}\right) + \lambda(t) p_{N0}$$

$$N(t) = \hat{U}(t,0) N_0 = N_0 + \int_0^t dt_1 \lambda(t_1)$$

$$a(t) = \left(\int_0^t N(t_1) dt_1 + a_0^3\right)^{\frac{1}{3}}$$

$$\phi(t) = \phi_0 + \sqrt{\frac{4}{3\kappa}} \ln\left(\frac{\int_0^t N(t_1) dt_1 + a_0^3}{a_0^3}\right)$$

Finite symmetry operator

$$\hat{S}(s) = \exp\left(s\{\quad ,G[\xi]\}_{PB}\right)$$

Parameter *s* labels one-parameter family of gauge transformed solutions associated with the finite group descriptors ξ

Note: could put *s* dependence in ξ to simplify coordinate transformations

Cosmology example: $$G[\xi] = \dot{\xi} p_N + \xi\left(-\frac{\kappa p_a^2}{24a} + \frac{p_\phi^2}{2a^3}\right)$$

$$N_s(t) = \hat{S}(s)N(t) = N(t) + s\dot{\xi}(t)$$

$$a_s(t) = \left(\int_0^t N(t_1)\,dt_1 + s\xi(t) + a_0^3 \right)^{\frac{1}{3}}$$

$$\phi_s(t) = \phi_0 + \sqrt{\frac{4}{3\kappa}} \ln\left(\frac{\int_0^t N(t_1)\,dt_1 + s\xi(t) + a_0^3}{a_0^3} \right)$$

5 - Gauge Fixing and Intrinsic Coordinates

- Claim: at least one gauge condition must be time-dependent
- Suggestion (dictated by necessity!): let physical fields fix the coordinates
- This was program proposed first by Einstein in reconciling himself with general covariance
  - See extensive analyses by John Stachel on Einstein's "hole argument"
- Komar and Bergmann proposed using Weyl scalars as intrinsic coordinates
- Only scalars may be used to fix intrinsic coordinates

Intrinsic coordinates

- If prescription to go to intrinsic coordinates is unique, all observers will agree on all values of geometric objects when they transform to this coordinate system
- These values are equivalently those obtained through the imposition of a gauge condition
- Indeed, the setting of coordinates equal to some function of the dynamical variables are gauge conditions

6 - Observables - Diffeomorphism Invariants

- We define an observable to be any dynamical quantity whose value is independent of the arbitrary choice of coordinates

- Observables are therefore defined to be functions of dynamical variables which are invariant under a change in coordinates

- The count of independent variables in invariant functions is just the number of degrees of freedom of the system
  - In GR this number is four per spatial location
  - for our cosmological model the number is two

Construction of invariants

We construct invariant phase space functions of the dynamical variables by gauge transforming solutions which do not satisfy the gauge condition to solutions which do

This fixes the symmetry group descriptor as the appropriate function of the original solution variables

Cosmology example (taking group parameter $s = 1$):

Let us pick the value of the scalar field as an intrinsic time,

So set

$$\phi_s(t) = T = \phi_0 + \sqrt{\frac{4}{3\kappa}} \ln\left(\frac{\int_0^t N(t_1)dt_1 + s\xi(t) + a_0^3}{a_0^3} \right)$$

Next solve for ξ as a function of the canonical variables

$$\xi(t) = -\int_0^t N(t_1)\,dt_1 - a_0^3 + a_0^3 \exp\left(\sqrt{\frac{3\kappa}{4}}(T - \phi_0)\right)$$

Use this descriptor to gauge transform all the remaining canonical variables:

$$\bar{a}(T) = a_0 \exp\left(\sqrt{\frac{\kappa}{12}}(T - \phi_0)\right)$$

$$\bar{N}(T) = \sqrt{\frac{3\kappa}{4}}\, a_0^3 \exp\left(\sqrt{\frac{3\kappa}{4}}(T - \phi_0)\right)$$

Magic Trick! How is it possible for variables to be invariant under Hamiltonian evolution and yet be invariant under diffeomorphisms?

Answer: There is no implicit time dependence on the canonical variables.

Cosmology example:

$$\bar{a}(t) = a_0 \exp\left(\sqrt{\frac{\kappa}{12}}(t - \phi_0)\right) = a(t)\exp\left(\sqrt{\frac{\kappa}{12}}(t - \phi(t))\right)$$

$$\bar{N}(t) = \sqrt{\frac{3\kappa}{4}}\, a_0^3 \exp\left(\sqrt{\frac{3\kappa}{4}}(t - \phi_0)\right) = \sqrt{\frac{3\kappa}{4}}\, a^3(t)\exp\left(\sqrt{\frac{3\kappa}{4}}(t - \phi(t))\right)$$

Demonstration of time-dependent invariants

Confirm that $\overline{N}(t)$ and $\overline{a}(t)$ are invariant under

$$\delta\phi(t) = \{\phi(t), G(\xi(t))\} = \frac{\xi(t)p_\phi}{a(t)^3}$$

$$\delta a(t) = \{a(t), G(\xi(t))\} = -\frac{\xi(t)\kappa p_a(t)}{12a(t)}$$

Austin College 1849

Note: the Poisson bracket relations satisfied by these invariant functions of phase variables are the relations satisfied by "starred" brackets of the Syracuse school, or equivalently Dirac brackets. No such group theoretical interpretation was available before the diffeomorphism-induced symmetry group was known and implementable!

8 - What about quantum gravity?

- There are practical difficulties in finding a generically monotonically increasing function of Weyl scalars for intrinsic clock, even just in a patch. Perhaps material fields could be used - or are required?
- Quantum time evolution can be given a sensible meaning
  - Improved Wheeler-DeWitt formalism?
  - Improved Hamilton-Jacobi approach?
- Want formalism in which lapse and shift are retained as quantum operators
  - Could attempt to solve constraints and gauge fixing
  - Group average over diffeomorphisms?

In praise of lapse and shift

- Retention of lapse and shift with full symmetry group means that if group can be implemented in quantum theory, conventional objection to canonical program that one is committed to a fixed foliation of spacetime is wrong!
- Full spacetime metric will be subject to quantum fluctuation
- Tools are available in connection approaches to construct surface measures with timelike components when timelike component of connection is retained (as it must be to implement symmetry group)
- Historical note: Bergmann school originally retained lapse and shift in canonical program (Bergmann and Anderson, 1950)

8 - Conclusions

- Canonical general relativity is covariant under symmetry transformations which are induced by the full four-dimensional diffeomorphism group
- Misunderstandings of the nature of this group have led to the mistaken conclusion that diffeomorphism invariants must be constant in time
- Similar misconceptions have led to the mistaken conclusion that the choice of a spacetime foliation leaves only the spatial diffeomorphism group as the remaining symmetry group
- There is good physical rationale for retaining the lapse and shift as classical and quantum variables. Indeed, they must be retained to exploit the full symmetry of general relativity
- Retention of gauge variables in loop quantum gravity may lead to quantization of time-like areas and volumes