Gauge fixing, observables, and the problem of time in general relativity

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Plan of Talk

1. Motivation
2. Projectability of diffeomorphism symmetries under Legendre map
3. Diffeomorphism-induced symmetry generators and Hamiltonian
4. Finite symmetry transformations and time evolution
5. Gauge fixing using intrinsic coordinates
6. Time-dependent diffeomorphism invariants
7. Quantum implications
8. Conclusions

1 - Motivation

- Desire to realize 4-D diffeomorphism symmetry in canonical approach to quantum gravity
- Lapse and shift should be quantum operators subject to quantum fluctuations
- We all know intuitively that “frozen time” is nonsense!

Collaborators and references

- "The issue of time in generally covariant theories and the Komar-Bergmann approach to observables in general relativity," (with J Pons) in preparation
- "The gauge group in the Ashtekar-Barbero formulation of canonical gravity,", in Proceedings of the Ninth Marcel Grossmann Meeting, edited by V.G. Gurzadyan, R. T. Jantzen and R. Ruffini, (World Scientific, New Jersey, 2002), 1298 (with J. Pons)
- "The gauge group and the reality conditions in Ashtekar's formulation of general relativity," Phys. Rev. **D62** , 064026 (2000) (with J.M. Pons and L.C. Shepley)
- "The gauge group in the real triad formulation of general relativity," Gen. Rel. Grav. **32**, 1727 (2000) (with J.M. Pons and L.C. Shepley)
- "Gauge transformations in Einstein-Yang-Mills theories," J. Math. Phys. **41**, 5557 (2000) (with J.M. Pons and L.C. Shepley)

2 - Legendre projectability of diffeomorphism symmetries

- All generally covariant models have singular Lagrangians

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) = 0$$

- Configuration-velocity functions which vary in direction of null directions are not projectable to phase space

$$\text{if} \quad \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \square^j = 0, \text{ then for } f(q,\dot{q}) \text{ to be projectable}$$

$$\text{it must satisfy } \square^j \frac{\partial f}{\partial \dot{q}^j} = 0$$

Proof - we mean by “projectable” that f is the pullback of a function *F(q,p)* on phase space:

$$\Box^j \frac{\partial f(q,\dot{q})}{\partial \dot{q}^j} = \frac{\partial F(q,p(q,\dot{q}))}{\partial p_k} \Box^j \frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^j} = 0$$

Relativistic free particle example

$$L = \frac{1}{2N}\dot{x}^2 \Box \frac{N}{2} \Box \quad \frac{\partial^2 L}{\partial \dot{q}\ \partial \dot{q}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0 \ \text{ since } \frac{\partial \mathrm{L}}{\partial \dot{\mathrm{q}}^5} \equiv \frac{\partial \mathrm{L}}{\partial \dot{N}} = 0$$

So projectable functions cannot depend on $\dot{N}$

- Consider variations of metric under infinitesimal coordinate transformations

$$x'^{\mu} = x^{\mu} - \delta x^{\mu}(x)$$

Contains time derivatives of lapse and shift

$$\delta g_{\mu\nu} = g_{\mu\nu,\rho}\,\delta x^{\rho} + g_{\mu\rho}\,\delta x^{\rho}_{,\nu} + g_{\rho\nu}\,\delta x^{\rho}_{,\mu}$$

where

lapse

shift

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + g_{cd}N^c N^d & g_{ac}N^c \\ g_{bc}N^c & g_{ab} \end{pmatrix}$$

Free particle example

Not Projectable

$$t' = t - \xi(t), \text{ so } \delta g_{00} = g_{00,0}\xi + 2g_{00}\dot{\xi} \Rightarrow \delta N = \dot{N}\xi + N\dot{\xi}$$

- Resolution: infinitesimal coordinate transformations must depend in a unique, precise way on the lapse and shift

$$\delta x^{\mu}(x) = \delta_a^{\mu}\xi^{a}(x) + n^{\mu}(x)\xi^{0}(x)$$

where

$$n^{\mu} = (N^{-1}, -N^{-1}N^{a})$$

is the normal to the constant time hypersurface

Free particle example:

$$t' = t - N^{-1}\xi \Rightarrow \delta N = \dot{N}N^{-1}\xi + N\frac{d}{dt}\left(N^{-1}\xi\right) = \dot{\xi}$$

3 - Symmetry Generators and Hamiltonian

Primary constraints

Secondary constraints

$$G[\Box] = \Box d^3x \left(\dot{\Box}^{\Box} P_{\Box} + \Box^{\Box}\left(\mathcal{H}_{\Box} + \Box\Box\Box d^3y\ d^3z C^{\Box}_{\Box\Box}(x,y,z) N^{\Box}(y) P_{\Box}(z)\right)\right)$$

Group structure functions: $\left\{\mathcal{H}_{\Box}(x), \mathcal{H}_{\Box}(y)\right\}_{PB} = \Box d^3z\ C^{\Box}_{\Box\Box}(x,y,z)\mathcal{H}_{\Box}(z)$

Free particle example:

Momentum conjugate to N

$$G[\Box] = \dot{\Box}\Box + \Box\frac{1}{2}\left(p^2 + 1\right)$$

Hamiltonian

Functions of dynamical canonical variables

$$H = \Box d^3x\left(N^{\Box}\mathcal{H}_{\Box} + \Box^{\Box}P_{\Box}\right)$$

Arbitrary functions of coordinates

Free particle example:

$$H = \frac{N}{2}\left(p^2 + 1\right) + \Box\Box$$

4 - Finite Time Evolution and Symmetry Transformations

Finite time evolution operator:

Time ordering

$$\hat{U}(t,0) = \mathcal{T}\exp\left[\int_0^t dt'\{\square, H(t')\}_{PB}\right]$$

$$= 1 + \int_0^t dt_1\{\square, H(t_1)\}_{PB} + \int_0^t dt_1 \int_0^{t_1} dt_2\{\{\square, H(t_1)\}_{PB}, H(t_2)\}_{PB} + \cdots$$

Free particle example:

$$H(t_1) = \frac{N(t_1)}{2}\left(p^2 + 1\right) + \square(t_1)\square$$

$$N(t) = \hat{U}(t,0)N = N + \int_0^t dt_1 \square(t_1)$$

$$x^{\square}(t) = \hat{U}(t,0)x^{\square} = x^{\square} + \int_0^t dt_1 N(t_1)p^{\square}$$

Finite symmetry operator

$$\hat{S}(s) = \exp\left(s\{\ \ ,G[\square]\}_{PB}\right)$$

Parameter s labels one-parameter family of gauge transformed solutions associated with the finite group descriptors $\square$

Note: could put s dependence in $\square$ to simplify coordinate transformations

Free particle example:

$$G[\square](t) = \frac{\square(t)}{2}\left(p^2 + 1\right) + \dot{\square}(t)\square$$

$$N_s(t) = \hat{S}(s)N(t) = N(t) + s\dot{\square}(t)$$

$$x_s{}^{\square}(t) = \hat{S}(s)x^{\square}(t) = x^{\square}(t) + s\square(t)p^{\square}$$

5 - Gauge Fixing and Intrinsic Coordinates

- Claim: at least one gauge condition must be time-dependent
- Suggestion (dictated by necessity!): let physical fields fix the coordinates
- This was program proposed first by Einstein in reconciling himself with general covariance
  - See extensive analyses by John Stachel on Einstein’s “hole argument”
- Komar and Bergmann proposed using Weyl scalars as intrinsic coordinates

Intrinsic coordinates

- If prescription to go to intrinsic coordinates is unique, all observers will agree on all values of geometric objects when they transform to this coordinate system
- These values are equivalently those obtained through the imposition of a gauge condition
- Indeed, the setting of coordinates equal to some function of the dynamical variables are gauge conditions

Free particle example: set $\bar{t} = f^{-1}(x^0(t))$ then

$$\bar{x}^a(\bar{t}) = x^a(t(\bar{t})) = x^a + \frac{p^a}{p^0}\left(f(\bar{t}) - x^0\right)$$

Free particle example: set $\bar{t} = f^{-1}(x^0(t))$ then

$$\bar{x}^a(\bar{t}) = x^a(t(\bar{t})) = x^a + \frac{p^a}{p^0}\left(f(\bar{t}) - x^0\right)$$

and

$$\bar{N}(\bar{t}) = N(t)\frac{dt}{d\bar{t}} = \frac{1}{p^0}\frac{df(\bar{t})}{d\bar{t}}$$

All observers agree on the form of these solution, regardless of the particular coordinates t with which they start

6 - Observables - Diffeomorphism Invariants

- We define an observable to be any dynamical quantity whose value is independent of the arbitrary choice of coordinates

- Observables are therefore defined to be functions of dynamical variables which are invariant under a change in coordinates

- The count of independent variables in invariant functions is just the number of degrees of freedom of the system
  - In GR this number is four per spatial location
  - for the free particle the number is six

Construction of invariants

We construct invariant phase space functions of the dynamical variables by gauge transforming solutions which do not satisfy the gauge condition to solutions which do

This fixes the symmetry group descriptor as the appropriate function of the original solution variables

Free particle example:

$$f(t) = x^0(t) + \square[x](t)p^0 \;\square\; \square[x](t) = \frac{1}{p^0}\left(f(t) \square x^0(t)\right)$$

$$x^a_{\square[x]}(t) = x^a(t) + \frac{p^a}{p^0}\left(f(t) \square x^0(t)\right) \qquad N_{\square[x]}(t) = N(t) + \dot{\square}[x](t) = \frac{1}{p^0}\frac{df(t)}{dt}$$

Demonstration of time-dependent invariants

Continuing with the free particle example, $x^0(t) = f(t)$ is invariant by construction. And it is time-dependent!

OK, you're not convinced. Fortunately, since we are now able to implement a canonical symmetry transformation we can check explicitly!

The non-vanishing infinitesimal variations generated by $G[\square](t)$ are

$$\square x^{\square} = \square(t)\, p^{\square}$$

Observe that f(t) doesn't depend on the phase space coordinates and is therefore trivially invariant!

Note that $\boxed{N_{\square[x]}(t) = \frac{1}{p^0}\frac{df(t)}{dt}}$ is invariant by the same by the same argument

Still not convinced?

Expressing our invariant functions from the last slide in terms of the phase space arguments we have

$$\boxed{x^a_{\square[x]}(t) = x^a(t) + \frac{p^a}{p^0}\left(f(t) \square x^0(t)\right) = x^a + \frac{p^a}{p^0}\left(f(t) \square x^0\right)}$$

so

$$\boxed{\square x^a_{\square[x]}(t) = \square x^a \square \frac{p^a}{p^0}\square x^0 = 0}$$

7 - What about quantum gravity?

- There are practical difficulties in finding a generically monotonically increasing function of Weyl scalars for intrinsic clock, even just in a patch. Perhaps material fields could be used - or are required?
- Quantum time evolution can be given a sensible meaning
  - Improved Wheeler-DeWitt formalism?
  - Improved Hamilton-Jacobi approach?
- Want formalism in which lapse and shift are retained as quantum operators
  - Could attempt to solve constraints and gauge fixing
  - Group average over diffeomorphisms?

In praise of lapse and shift

- Retention of lapse and shift with full symmetry group means that if group can be implemented in quantum theory, conventional objection to canonical program that one is committed to a fixed foliation of spacetime is wrong!
- Full spacetime metric will be subject to quantum fluctuation
- Tools are available in connection approaches to construct surface measures with timelike components when timelike component of connection is retained (as it must be to implement symmetry group)

Quantum lapse of relativistic free particle

The lapse in our our free particle model is readily promoted to an operator with a well-defined physical meaning - the proper time of the particle is subject to quantum fluctuation!

It is assumed that Minkowski observers have rate adjusted their clocks, as instructed, with the intrinsic time choice $x^0(t) = f(t)$

The proper time elapse between t_i and t_f is

$$\Delta\tau = \int_{t_i}^{t_f} dt \frac{df(t)}{dt} \frac{1}{\hat{p}^0} = \left(f(t_f) - f(t_i) \right) \frac{1}{\sqrt{\hat{\vec{p}}^2 + 1}}$$

So the smaller the uncertainty in particle spatial momentum (and energy) , the larger the uncertainty in the proper time!

8 - Conclusions

- Canonical general relativity is covariant under symmetry transformations which are induced by the full four-dimensional diffeomorphism group
- Misunderstandings of the nature of this group have led to the mistaken conclusion that diffeomorphism invariants must be constant in time
- Similar misconceptions have led to the mistaken conclusion that the choice of a spacetime foliation leaves only the spatial diffeomorphism group as the remaining symmetry group
- There is good physical rationale for retaining the lapse and shift as classical and quantum variables. Indeed, they must be retained to exploit the full symmetry of general relativity