It still not universally recognized that all the gauge symmetries present in
the Lagrangian formalism for a physical theory must also manifest themselves in
the canonical formalism. This issue has been particularly obscure and controver-
sial in the case of generally covariant theories, where the full realization of the
diffeomorphisms-induced gauge group in phase space has met with the difficulty of
the lack of projectability of some generators from the configuration-velocity space
to phase space.

In a collaboration that started some time ago, we have tried to clarify the dif-
ferent issues involved, particularly that of projectability.\textsuperscript{1,2,3,4} We have obtained
the general result that, in generally covariant theories with a metric, the arbitrary
functions $\delta x^a$ that describe an infinitesimal diffeomorphism in spacetime must con-
tain specific and compulsory dependences on the lapse and shift gauge variables
in order for the transformations of the fields be projectable to phase space. If a
Yang-Mills type field is included and the gauge symmetry is larger than that of the
spacetime diffeomorphisms, other dependences with respect to other gauge vari-
ables may become necessary in order to ensure projectability. In turn, the fact that
arbitrary functions of the gauge transformations explicitly depend on some fields
makes the structure “constants” associated with the algebra of generators no longer
constants, but functions of the fields. Since the gauge group, acting on the space of
field configurations, contains the transformations induced by diffeomorphisms ac-
ting on spacetime, the fact that the diffeomorphism group by itself is not realizable
in phase space is compatible with the fact that a wise selection of the generators of
the gauge group can make it fully realizable in phase space.

The application of this program to the Ashtekar formulation of complex canoni-
cal relativity has, besides the requirements of projectability the additional subtlety
that the reality conditions need to be preserved. In fact, we find that both aspects
are interrelated.\textsuperscript{4}

Our presentation here concerns the extension of our previous work to the case of
the Ashtekar-Barbero real connection or, even more generally, to connection-based
models described by an arbitrary value of the Immirzi-Barbero parameter.

Our results, showing how the diffeomorphisms-induced gauge group is fully im-
plemented in phase space, suggest the possibility of applying new techniques to the
loop quantum gravity program; it is possible in principle to retain all the variables
in the formalism—not simply eliminating the gauge variables as dynamical objects through the imposition of gauge conditions. In retaining the gauge variables we also preserve the full content of the gauge group. This possibility may prove to be relevant for a quantization which seeks true spacetime diffeomorphism invariants.

Here we report some of our results. Our connection variable is defined as

$$A^i_a = \omega^i_a - \alpha^{-1} K^i_a$$

where $$\omega^i_a$$ are the spatial parts of the Ricci rotation coefficients (spin connections), and $$K^i_a$$ is on shell the extrinsic curvature of the equal time surfaces contracted with the triad fields. Indices $$a$$ correspond to the space coordinates in spacetime whereas indices $$i$$ are "at space" indices. $$\alpha$$ is an arbitrary parameter—essentially the Immirzi-Barbero parameter—and for $$\alpha = \sqrt{-1}$$ we recover the original Ashtekar formulation. The fundamental Poisson bracket is

$$\{ T^q_i, A^j_b \} = \alpha^{-1} \delta^q_i (x - x') \delta^a_b \delta^j_b$$

where $$T^q_i$$ is the densitized triad. The canonical Hamiltonian is a linear combination of secondary first class constraints,

$$\mathcal{H}_c = -\alpha A^i_0 \mathcal{H}_i + N^a \mathcal{H}_a + N^a \mathcal{B}_a$$

where the lapse and shift functions are exhibited and $$\alpha A^i_0 = \Omega^i_0 - \alpha^{-1} K^i_a N^a + \alpha T^i_a N_a$$. The secondary constraints, all first class, are

$$\mathcal{H}_i = -\alpha D_a T^a_i + \mathcal{H}_a = \alpha T^a_i F^i_{ba}$$

and $$\mathcal{H}_0 = -\frac{1}{2} T^a_i T^b_j (-\alpha F^i_{ab} + (1 + \alpha^2 \rho^i_{ab})$$,

where $$\rho^i_{ab}$$ and $$\rho^i_{ab}$$ are the curvature tensors associated with the Ashtekar-Barbero connection and the spin coefficient, respectively, and $$\alpha D_a$$ is the covariant derivative corresponding to the connection $$\alpha A^i_a$$.

The primary constraints are the variables canonically conjugate to the lapse, shift, and time component of the Ricci rotation coefficient. Combination of primary and secondary constraints, all first class, allows one to construct the full set of gauge generators according to the expression

$$G[\xi] = P_A \dot{\xi}^A + (\mathcal{H}_A + \mathcal{P}_C \alpha N^B C_{AB}^C) \xi^A$$

where $$C_{AB}^C$$ are the structure functions associated with the Poisson brackets of the secondary constraints and the descriptors $$\xi$$ are infinitesimal arbitrary functions of spacetime. We have demonstrated that the variations produced by these generators are precisely those produced by those combinations of field dependent spacetime diffeomorphisms and the Yang-Mills-like transformations of the Ashtekar-Barbero connections which result in projectable configuration-velocity space variations. This proves that the gauge group is fully realized in phase space. Finally, we report that it is possible to preserve reality conditions when the parameter $$\alpha$$ takes any complex value. A full exposition of these results will appear elsewhere.

References